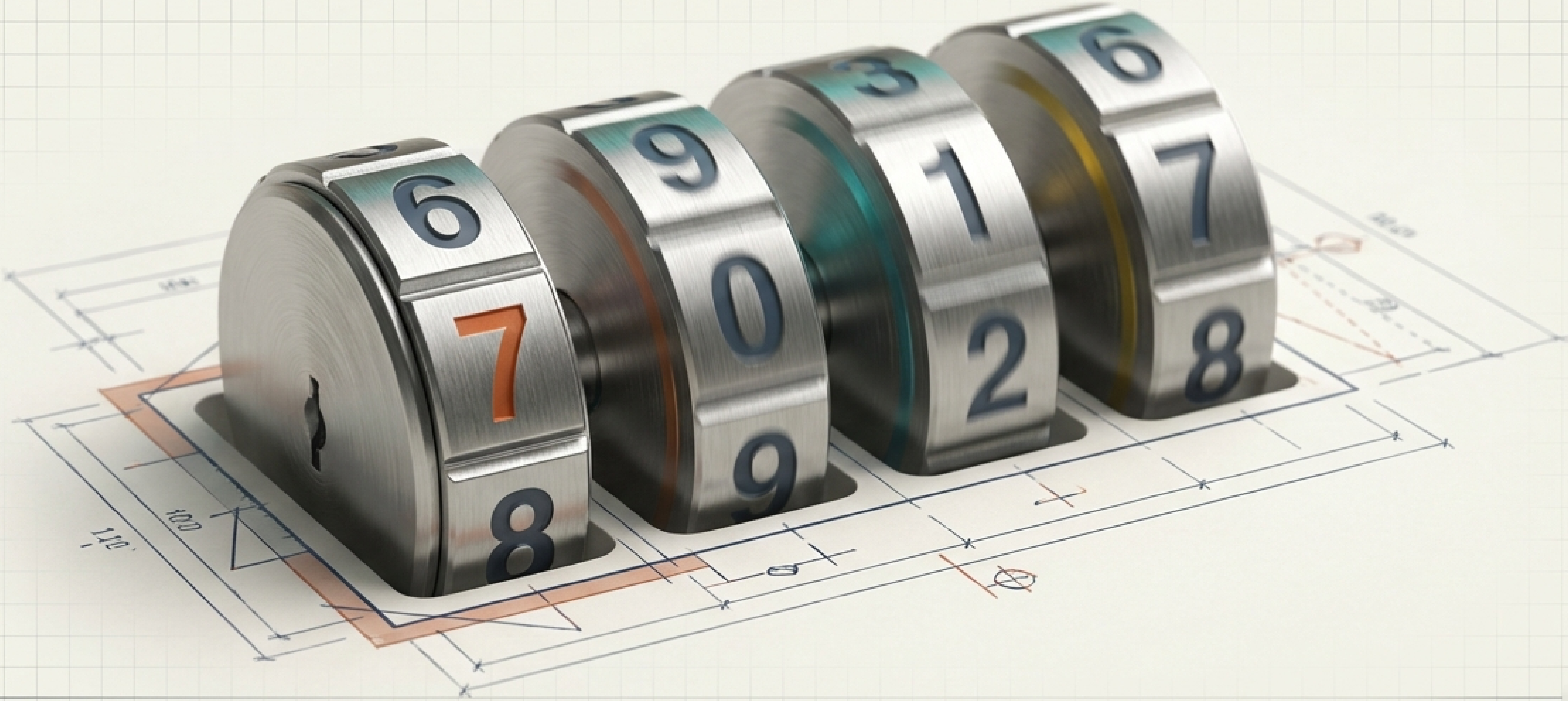
The Mathematics of Arrangement

A Visual Guide to Counting, Factorials, and Permutations



The Combinatorial Game Board

THE SETUP

You have a 4-wheel suitcase number lock with digits 0 through 9.

THE CONSTRAINT

No digit can be repeated. You remember the first digit is 7.

THE PROBLEM

How many 3-digit sequences are left to check?

[7] × [?] × [?] × [?]

Brute-force counting is tedious. We need a mathematical toolkit to calculate possibilities without listing them.

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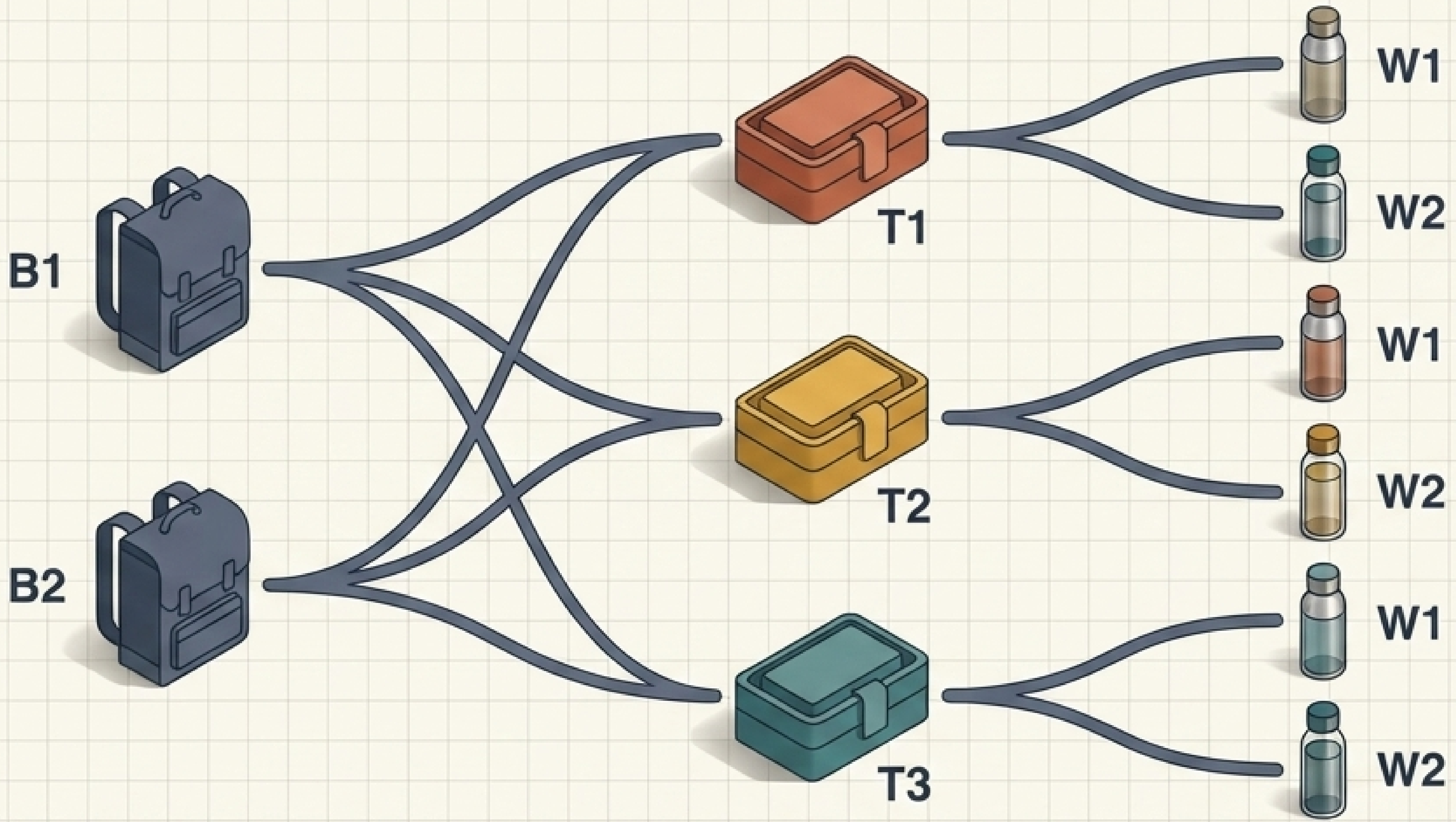
The Fundamental Principle of Counting



$$2 \times 3 \times 2 = 12$$

distinct pathways

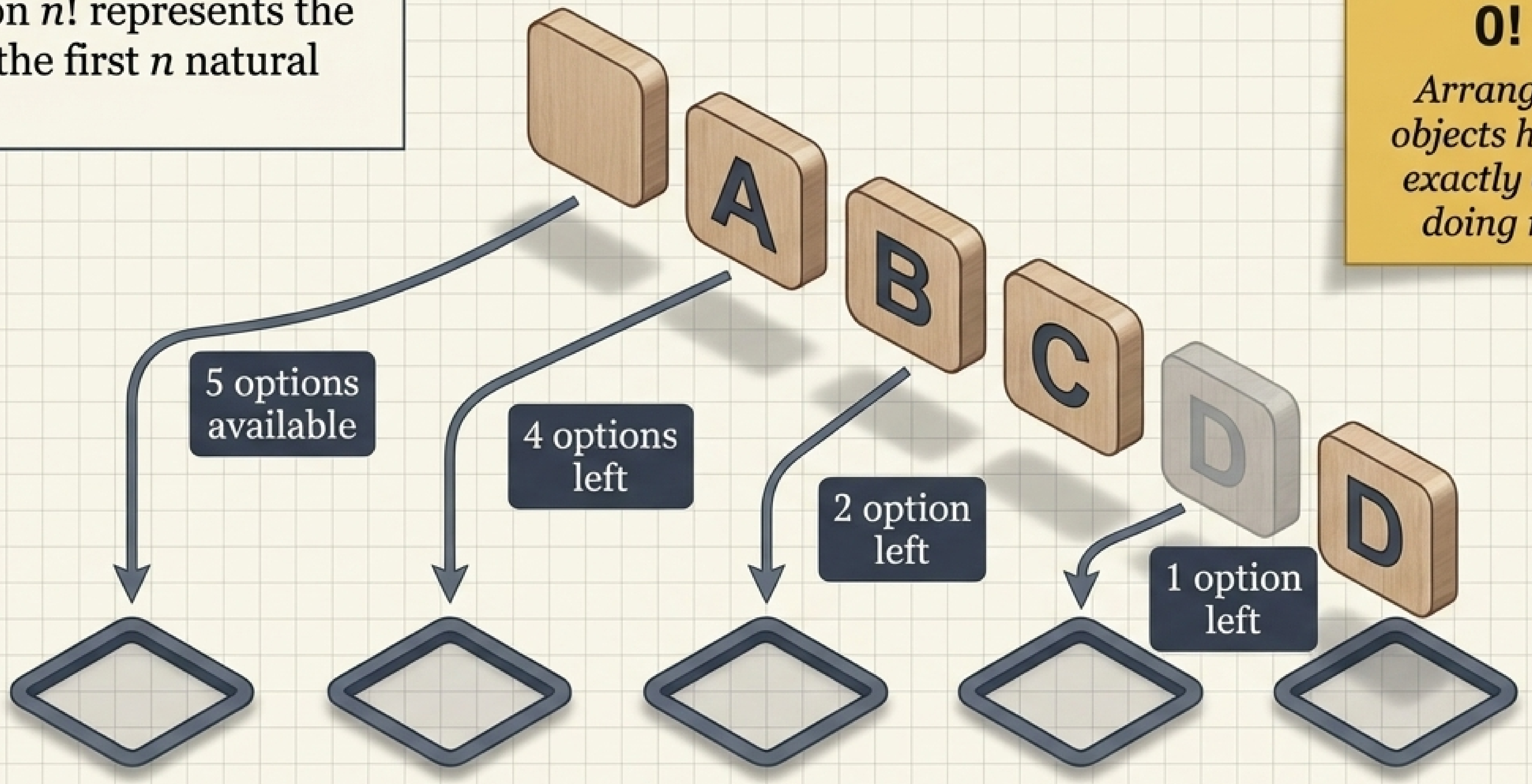
If an event can occur in m ways, and another in n ways, the total occurrences happen in $m \times n$ ways.
Every complex arrangement is just a sequence of simple choices.

The Language of Scale: Factorials (n!)

The notation $n!$ represents the product of the first n natural numbers.

Axiom:
$0! = 1$

Arranging zero objects happens in exactly one way: doing nothing.



$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Permutations: When Order Matters

Top Panel

Using all objects:

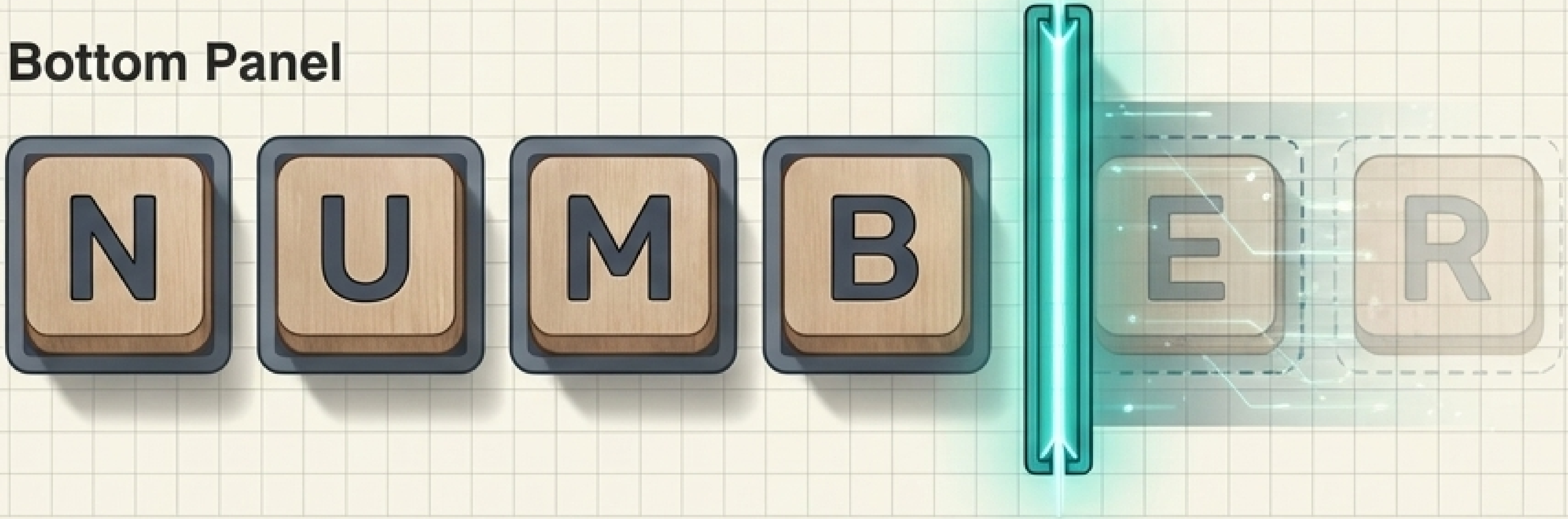
$$4! = 24$$

The Formula:

$${}_nP_r = \frac{n!}{(n-r)!}$$

The denominator $(n-r)!$ simply cancels out the unused tail of the factorial.

Bottom Panel



Arranging 3 letters from a pool of 6:

$$6 \times 5 \times 4 = 120$$

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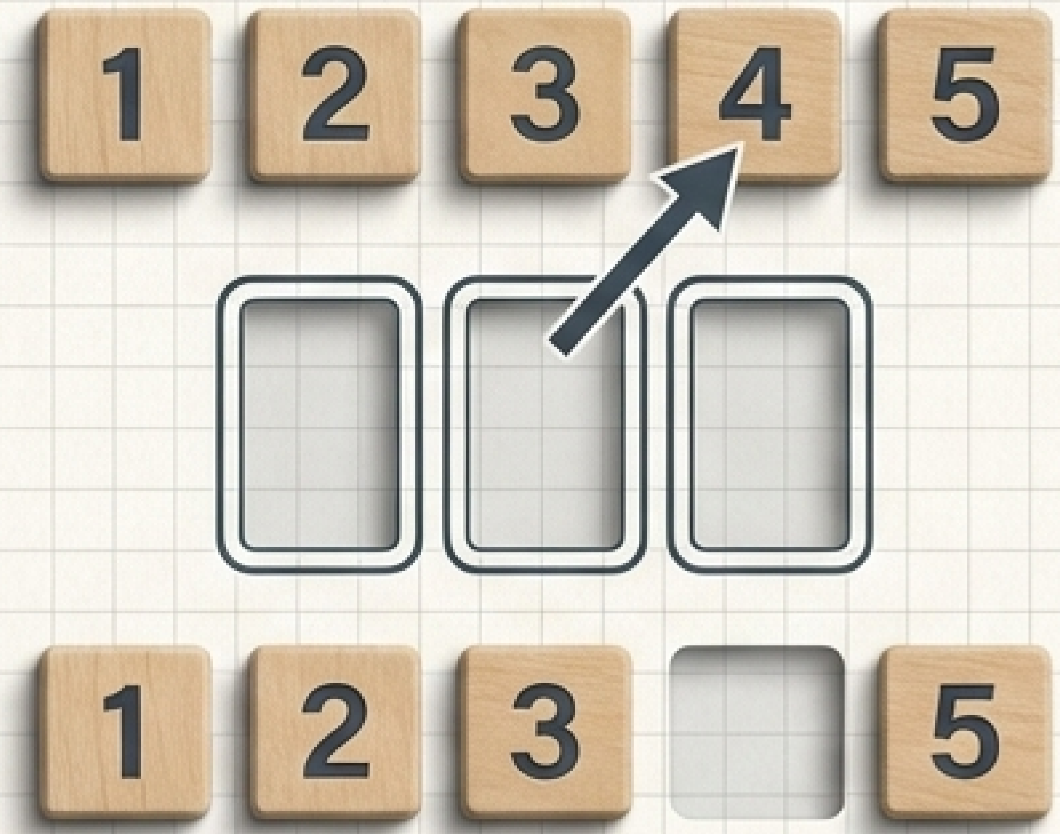
The "Repetition Allowed" Rule

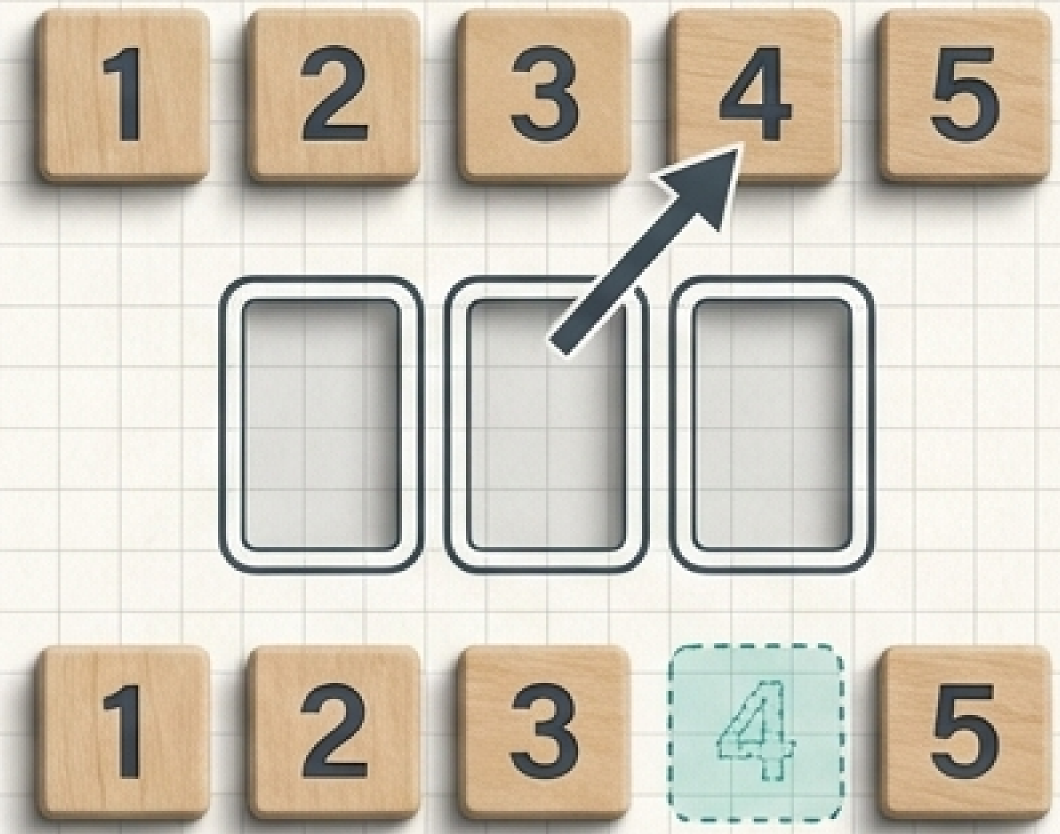
No Repetition



$$5 \times 4 \times 3 = 60$$

Source pool shrinks with each selection.

Repetition Allowed

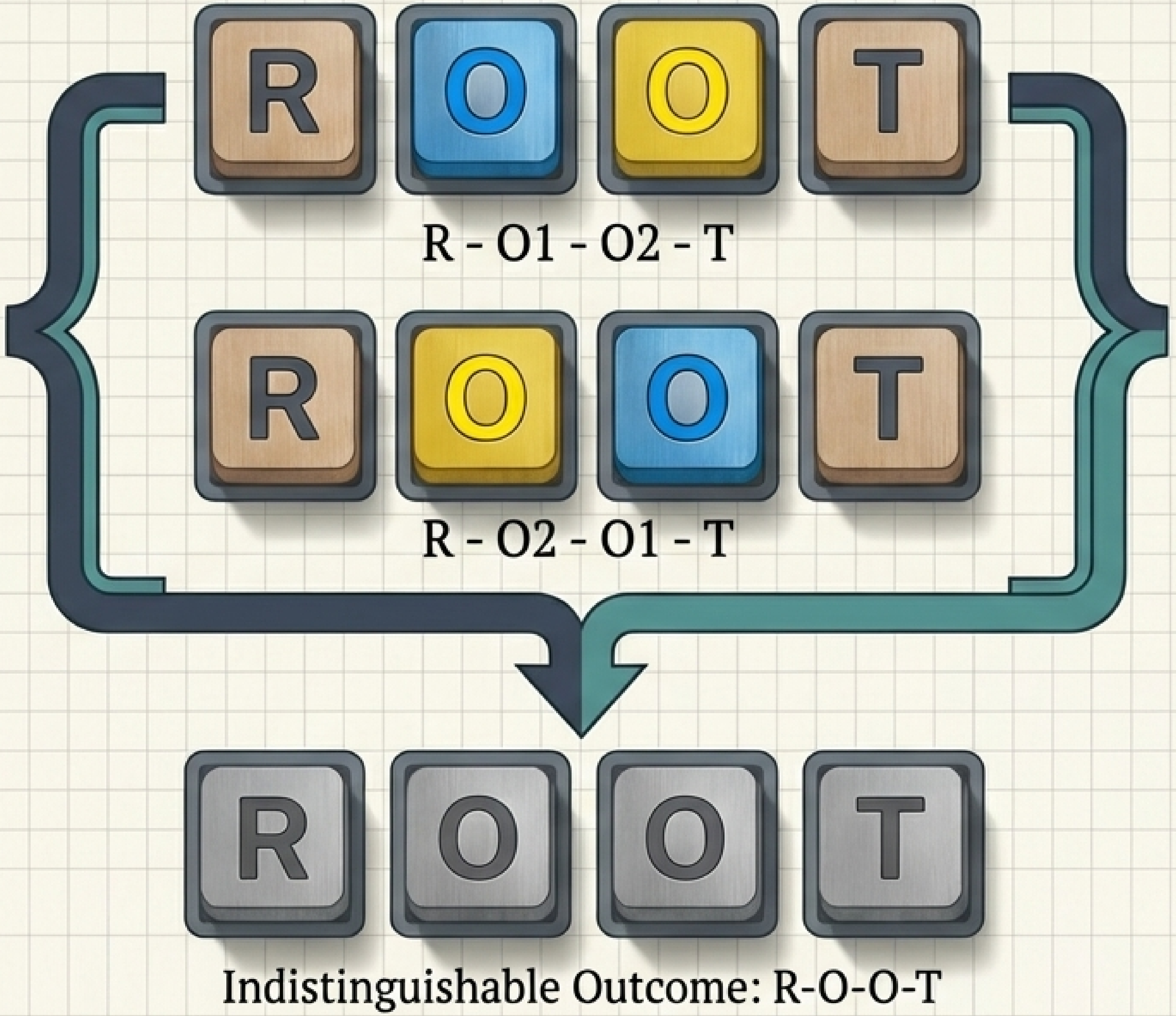


$$5 \times 5 \times 5 = 125$$

A new copy replaces the selected item.

Takeaway: When objects can be reused, the total number of arrangements of n objects in r slots is simply n^r

The Duplicate Dilemma



- Raw permutations treating O's as distinct:

  $4! = 24$

- Because the two identical O's can swap places in 2! ways without changing the word, we must divide to remove the ghost duplicates.

- True permutations:

  $4! / 2! = 12$

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Scaling the Master Formula for Duplicates

A L L A H A B A D

Total Tiles

$$\frac{9!}{4!\;\;2!} =$$

7,560 unique arrangements

Gold A's

Teal L's

Rule: Divide the total factorial by the factorial of each set of identical items.

Constraint Bundling: The ‘Together’ Tactic

Solving for specific grouping restrictions.

1. Identify and Group

2. Treat as a Single Super-Tile

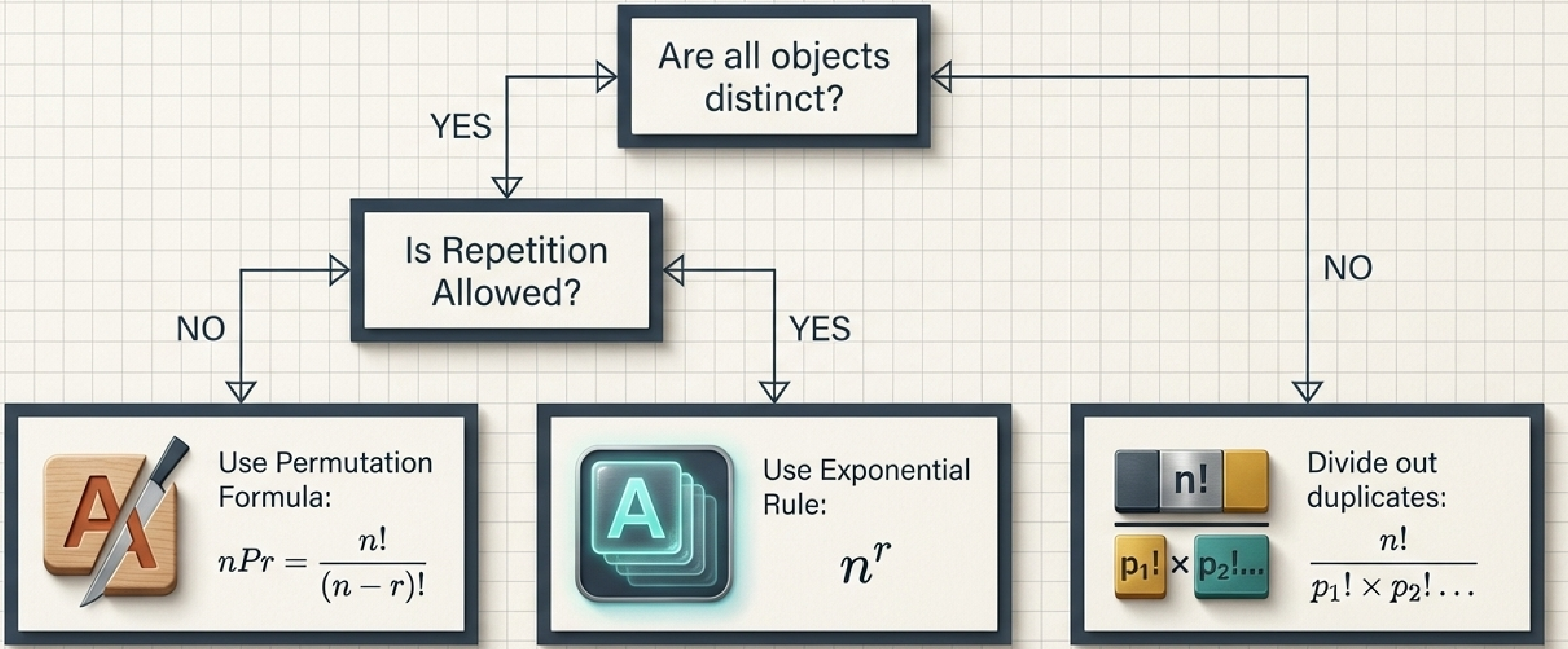
Arrange 6 outer objects: 6! ways

3. Account for Internal Shuffling

Internal arrangement: 3! ways

Calculation: $6! \times 3! = 720 \times 6 = 4{,}320$

The Permutation Diagnostic Matrix



Master the constraints, and you master the arrangement.