Complex Numbers

Expanding the Mathematical Universe

The Boundary of Real Numbers

The Limitation

$$x^2 + 1 = 0$$

$$\downarrow$$

$$x^2 = -1$$

The square of every real number is non-negative. In the real number system, this equation has no solution.

The Key

$$i = \sqrt{-1}$$

$$\downarrow$$

$$\text{Therefore, } i^2 = -1$$

The imaginary unit *i* is the exact key needed to extend the real number system into a larger, functional universe.

The Architecture of z

Side Note

Equality Rule:
Two complex numbers ($z_1 = a + ib$ and $z_2 = c + id$) are perfectly equal if and only if $a = c$ AND $b = d$.

The Imaginary Unit

$$z = \underbrace{a}_{\text{Real Part (Re z)}} + \overset{\text{The Imaginary Unit}}{i}\underbrace{b}_{\text{Imaginary Part (Im z)}}$$

Real Part (Re z)

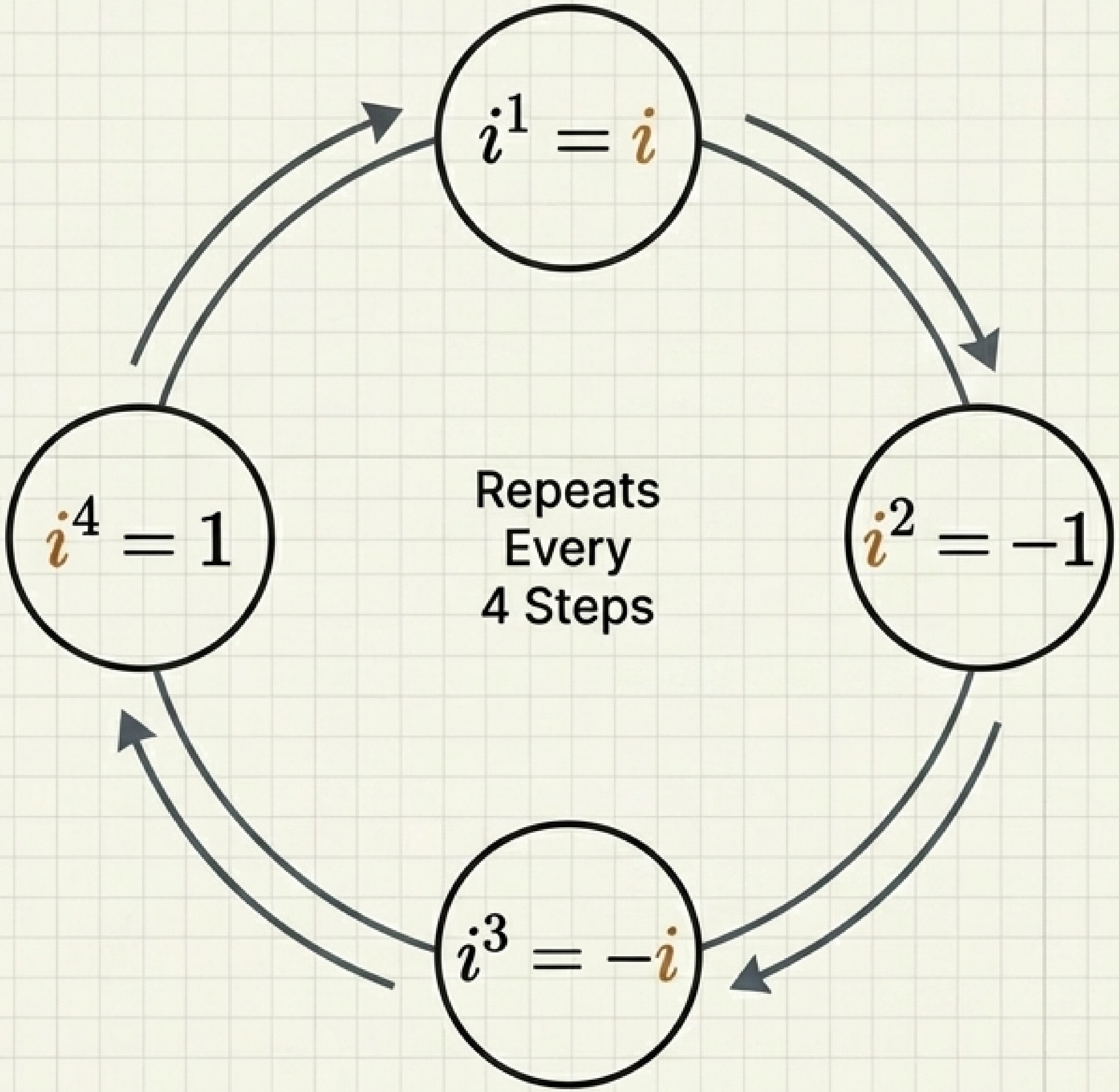
Imaginary Part (Im z)

Example: $z = 2 + 5i$

Re z = 2

Im z = 5

The Cyclical Power of i



The Universal Integer Rule

For any integer k:

$$i^{4k} = 1$$

$$i^{4k+1} = i$$

$$i^{4k+2} = -1$$

$$i^{4k+3} = -i$$

Linear Algebra: Like Interacts with Like

Addition	Subtraction
$(a + ib) + (c + id) = \underbrace{(a + c)} + i\underbrace{(b + d)}$	$(a + ib) - (c + id) = (a - c) + i(b - d)$
Group real with real, imaginary with imaginary.	The difference operates on identical axes.
$(2 + 3i) + (-6 + 5i) = -4 + 8i$	$(6 + 3i) - (2 - i) = 4 + 4i$

Additive Inverse: The negative of $z = a + ib$ is purely inverted: $-z = -a - ib$.

Complex Algebra: Cross-Multiplication and Division

Multiplication

$$z_1 z_2 = (ac - bd) + i(ad + bc)$$

$$(a + ib)(c + id)$$

$$ac + iad + ibc + i^2bd$$

i^2 becomes **-1**, flipping the sign to **-bd**

Division & Inverses

Multiplicative Inverse (z^{-1})

$$z^{-1} = \frac{1}{z} = \frac{a - ib}{a^2 + b^2}$$

Division (z_1/z_2)

$$\frac{z_1}{z_2} \rightarrow \frac{z_1}{z_2} \cdot \frac{\text{conjugate of } a - ib}{\text{conjugate of } a - ib} \rightarrow$$

Multiply by the denominator's conjugate to remove i from the bottom.

Measuring and Reflecting: Modulus & Conjugate

Modulus $|z|$

$$|z| = \sqrt{a^2 + b^2}$$

The non-negative real number representing the magnitude of z.

$$|2 - 3i| = \sqrt{(2^2 + (-3)^2} = \sqrt{13}$$

The Golden Identity

$$z * \bar{z} = |z|^2$$

Multiplying a complex number by its conjugate always yields a purely real number (the square of its modulus).

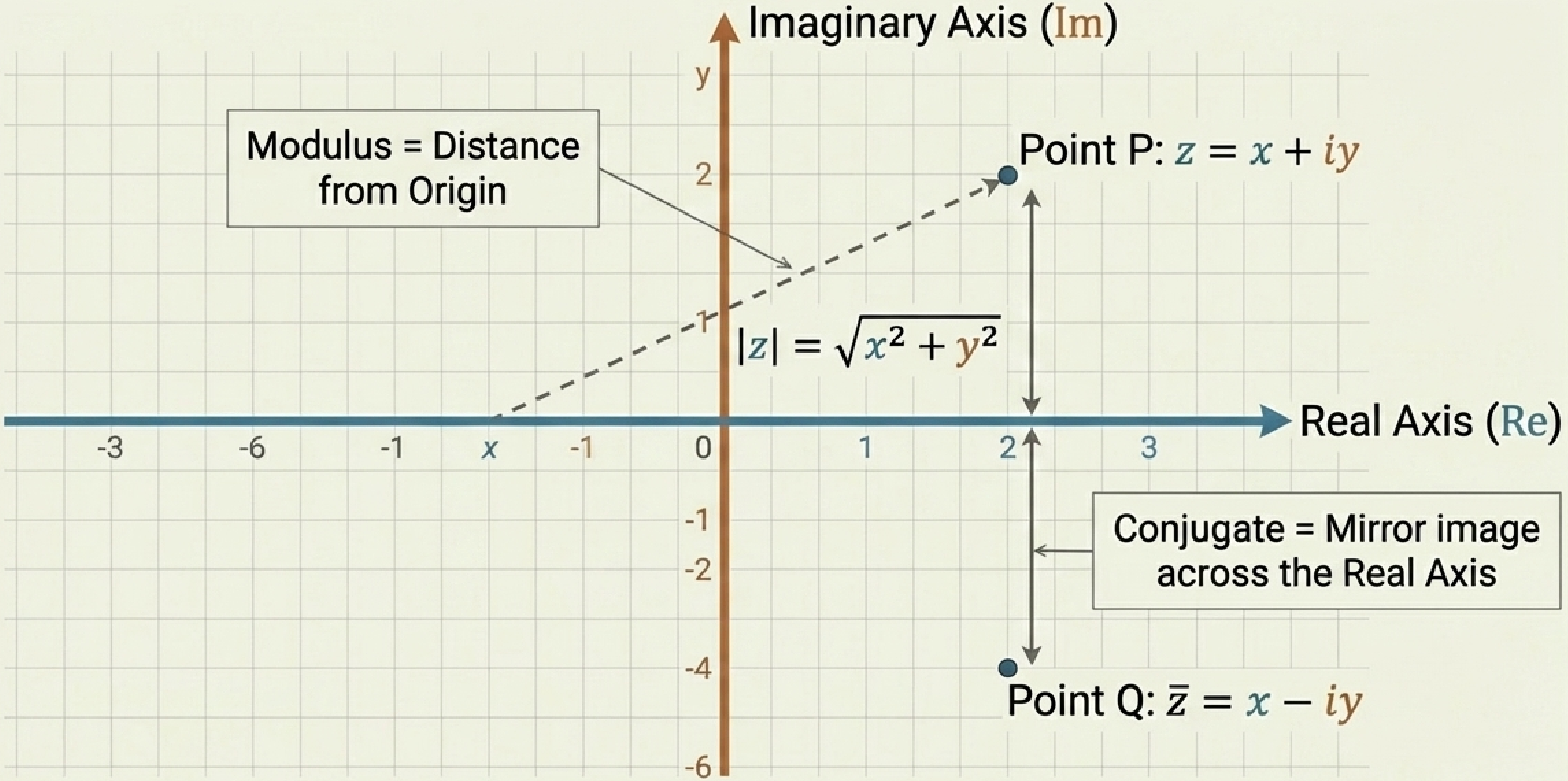
Conjugate $\bar{z}$

$$\bar{z} = a - ib$$

The complex number with an inverted imaginary part.

Conjugate of $2 + 3i = 2 - 3i$

The Argand Plane: Algebra Becomes Geometry



The Complex System Synthesis

The Blueprint

$$z = a + ib$$

(a = Real, b = Imaginary)

Equality:

$z_1 = z_2$ only if
$a = c$ and $b = d$

The Physics (Rules & Ops)

The i Cycle:

$$i = \sqrt{-1},\ i^2 = -1,$$
$$i^3 = -i,\ i^4 = 1$$

Arithmetic:

$$z_1 + z_2 = (a+c) + i(b+d)$$
$$z_1 z_2 = (ac-bd) + i(ad+bc)$$

The Map (Identities)

Modulus:

$$|z| = \sqrt{a^2 + b^2}$$

Conjugate:

$$\overline{z} = a - ib$$

Master Identity:

$$z * \overline{z} = |z|^2$$

Inverse:

$$z^{-1} = \frac{\overline{z}}{|z|^2}$$