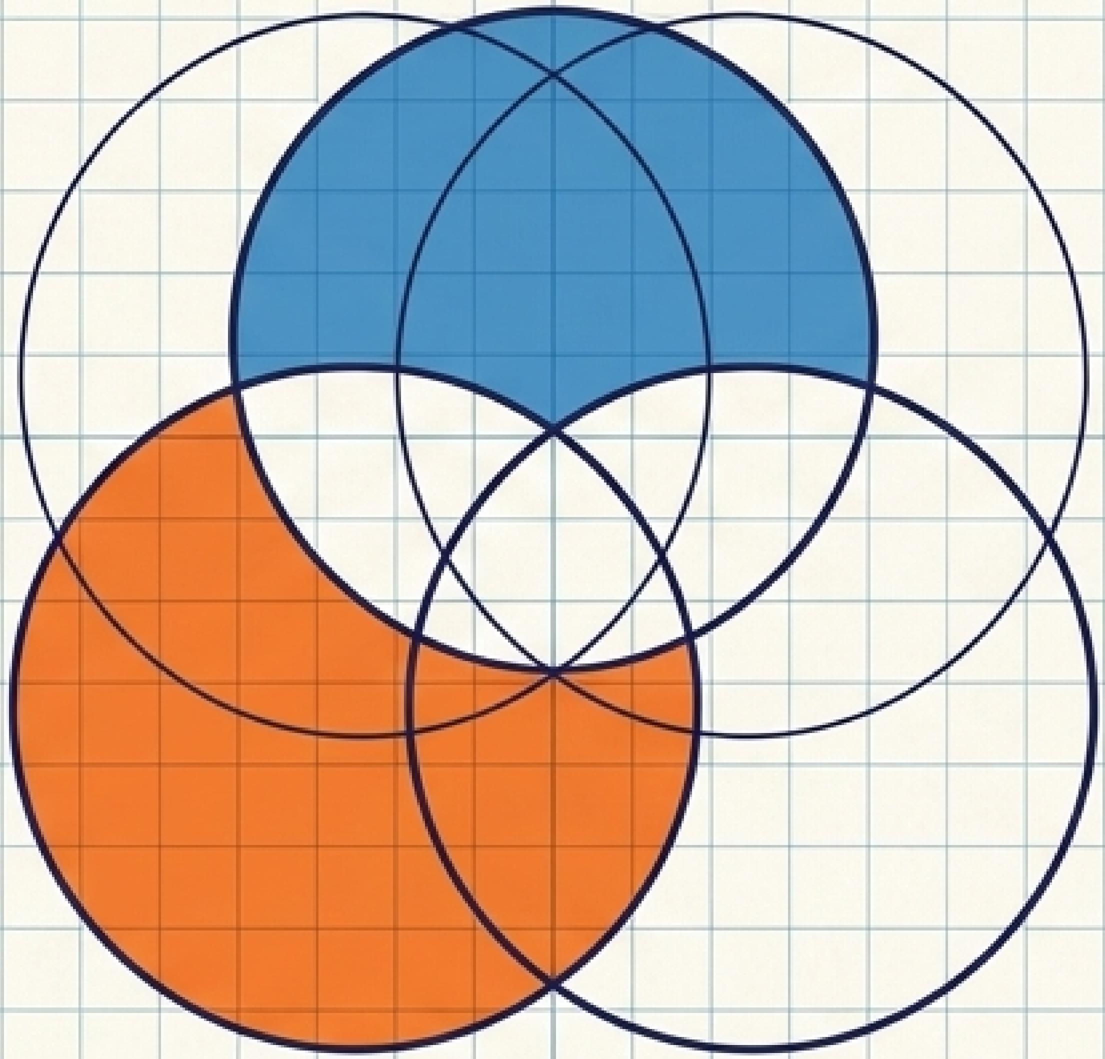
The Architecture of Logic

A Visual Guide to Set Theory & Mathematical Collections



"A study which did not begin with Pythagoras and will not end with Einstein; but is the oldest and the youngest. — G.H. Hardy

Originating with Georg Cantor (1845-1918) as a solution to trigonometric series, Set Theory is now the structural foundation of modern mathematics.

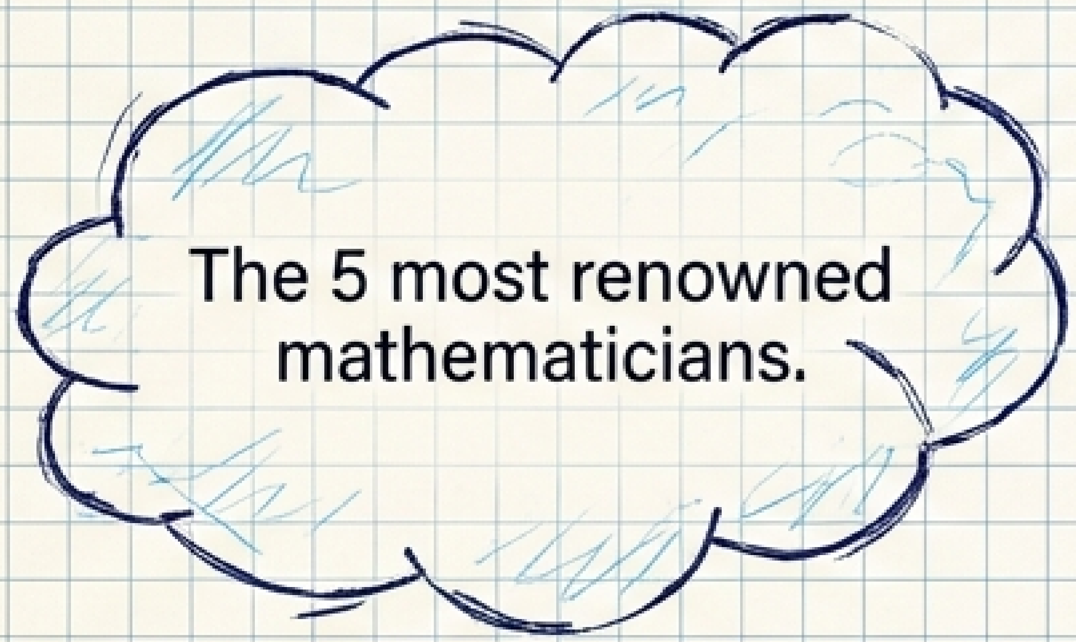
The Fundamental Unit of Mathematics

STRUCTURAL CALLOUT

A **set** is not just a collection; it is a **well-defined collection**. You must be able to **definitively decide** if a specific object belongs inside the **boundary** or not.

STRUCTURAL LABELS

The Undefined Cloud



STRUCTURAL WARNING:

Subjective boundary.
Varies from person to person.
NOT A SET.

STRUCTURAL LABELS

The Precision Boundary

Odd natural numbers less than 10.

$\boxed{1,\ 3,\ 5,\ 7,\ 9}$

APPROVAL LABEL:

Objective boundary.
1, 3, 5, 7, 9 definitively belong.
VALID SET.

Notation Callout:

$a \in A$: Belongs to (e.g., $3 \in$ *Odd natural numbers* < 10)

$b \notin A$: Does not belong (e.g., $2 \notin$ *Odd natural numbers* < 10)

Representing the Data: Two Languages

Roster Form (The Explicit List)

All elements are explicitly listed, separated by commas, enclosed in braces. Order is immaterial.

$$V = \{a, e, i, o, u\}$$

Set-Builder Form (The Algorithmic Rule)

Describes the single common property uniting the elements.

$$V = \{x : x \text{ is a vowel}\}$$

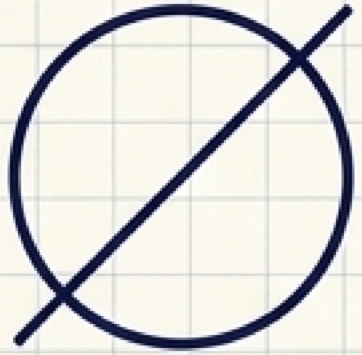
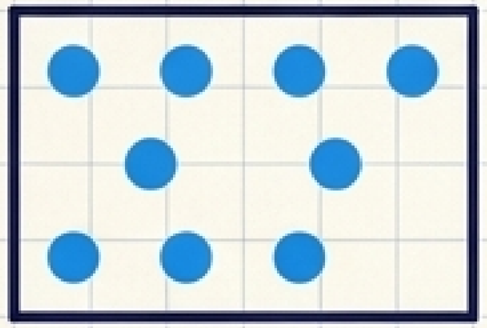
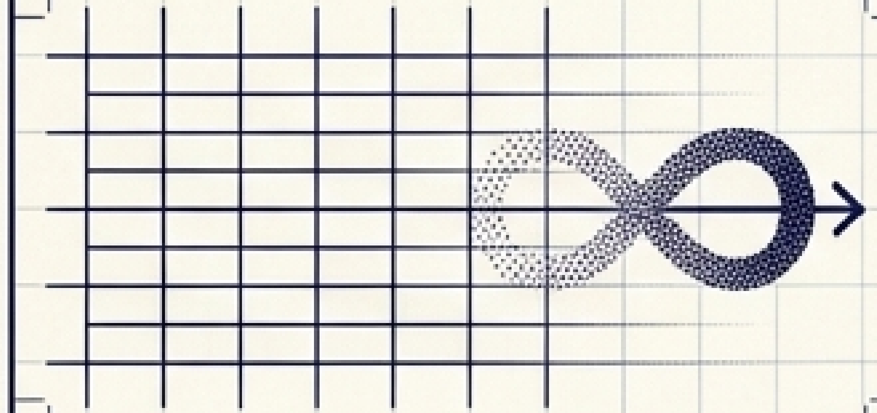
Anatomy of Syntax

3. Such that

$$V = \{x : x \text{ is a vowel}\}$$

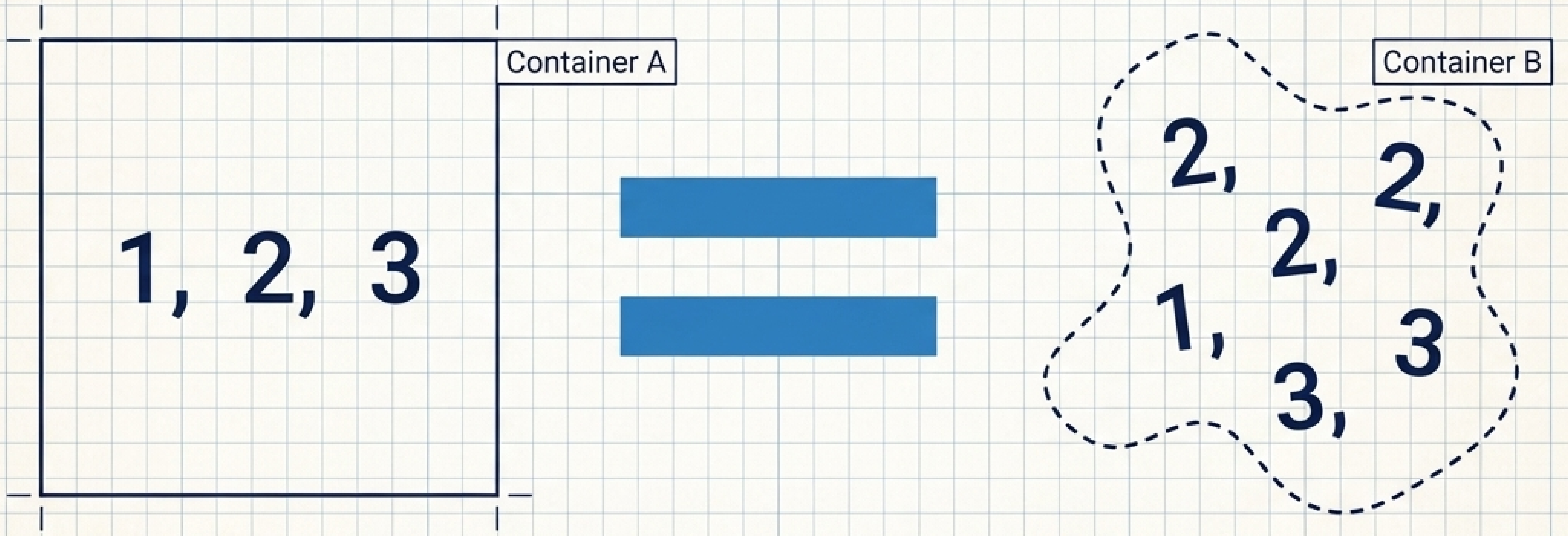
1. The set of all
2. The element variable

The Typology of Sets: Capacity and Limits

	**The Empty Set** (∅ or { })	**Definition:** Contains zero elements (null/void set). **Example:** $C = \{x : x \text{ is an even prime number} > 2\}$
	Finite Sets	**Definition:** Consists of a definite, countable number of distinct elements, denoted by $n(S)$. **Example:** W = The days of the week. ($n(W) = 7$)
	Infinite Sets	**Definition:** Elements continue indefinitely without following a bounded pattern. **Example:** G = The set of points on a line.

The Rule of Equality: Contents Over Form

Core Insight: Two sets are equal $(A = B)$ if and only if they have exactly the same elements. The shape, order, and repetition of the data do not change the fundamental set.

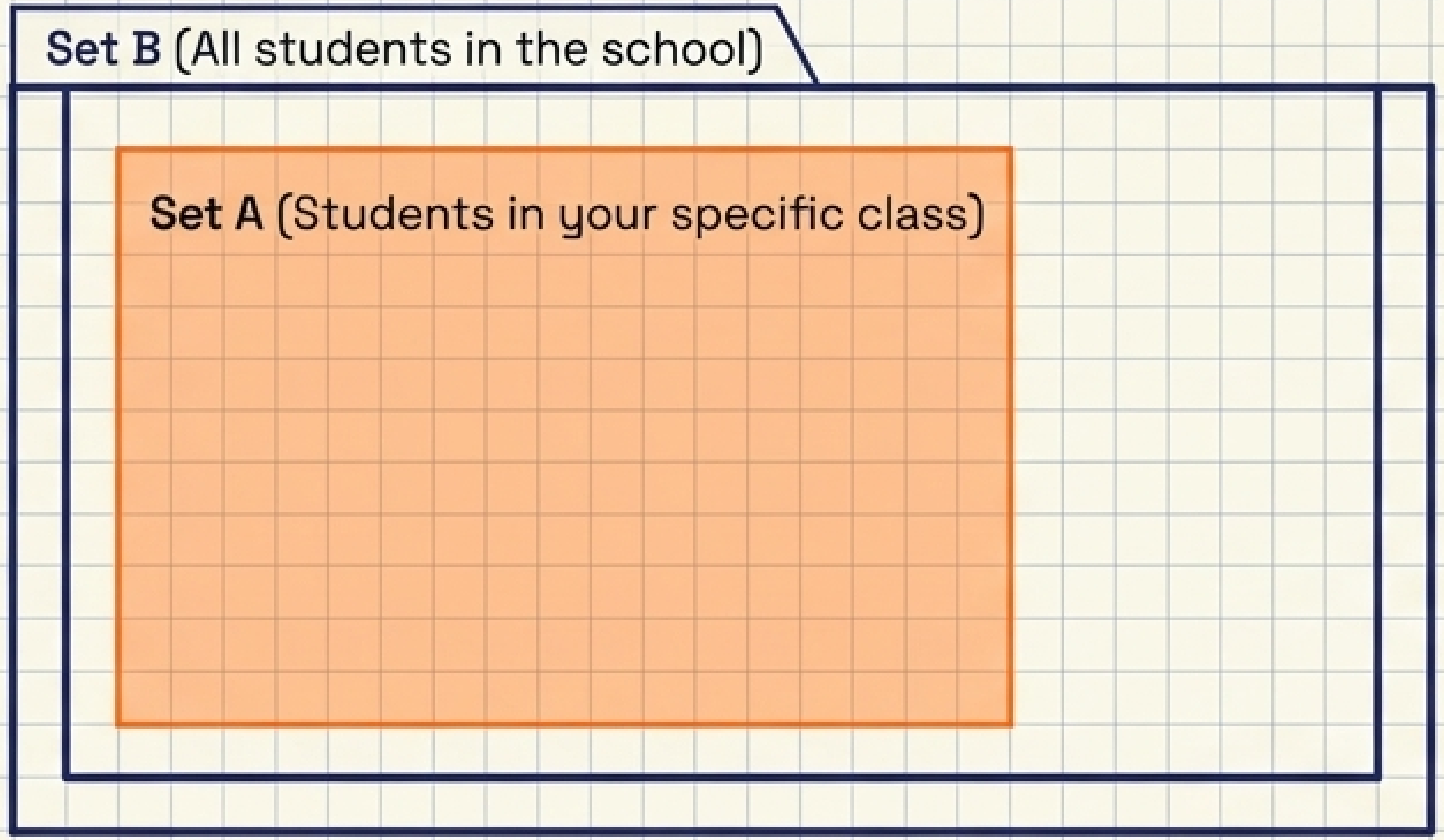


Repetition and order are mathematically irrelevant. Every element in A is in B, and every element in B is in A. Therefore, $A = B$.

Subsets: The Logic of Containment

If every element of set A is also an element of set B, then A is a subset of B.

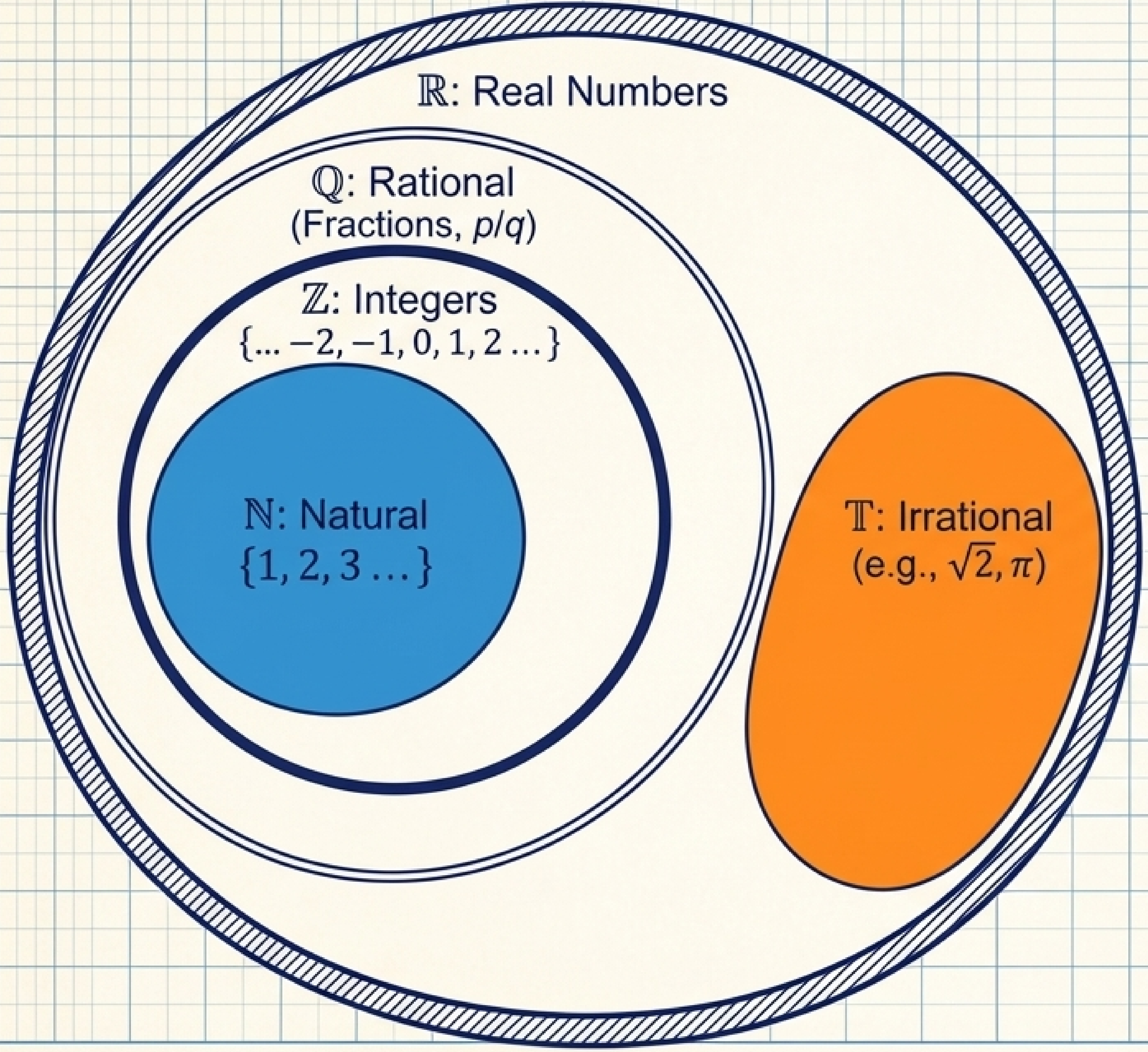
Notation: $A \subset B$
(Whenever $a \in A \Rightarrow a \in B$)



Edge Cases

1. Self-Containment	2. The Absolute Minimum	3. Proper vs. Superset
Every set is a subset of itself ($A \subset A$).	The empty set is a subset of every set ($\varnothing \subset A$).	If $A \subset B$ and $A \neq B$, A is a proper subset, and B is the superset.

The Hierarchy of Human Numbers



The Containment Rule:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

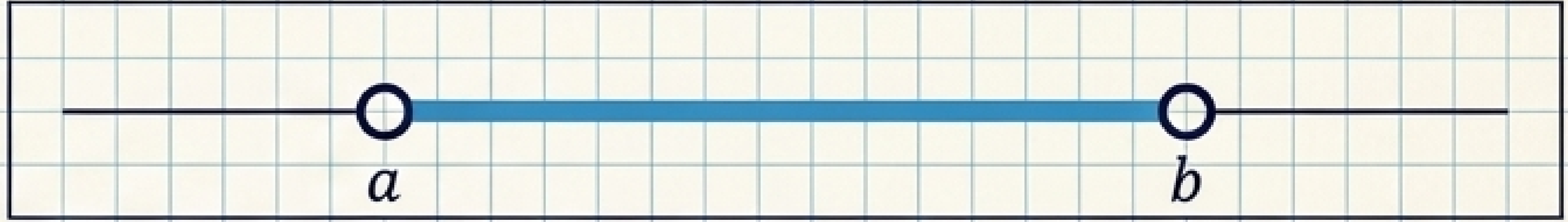
The Outliers:
Irrational numbers ($\mathbb{T}$) sit independently within the Real ($\mathbb{R}$) boundary, disjoint from Rationals ($\mathbb{Q}$).

Note: $\mathbb{T} \subset \mathbb{R}$, but $\mathbb{N} \not\subset \mathbb{T}$.

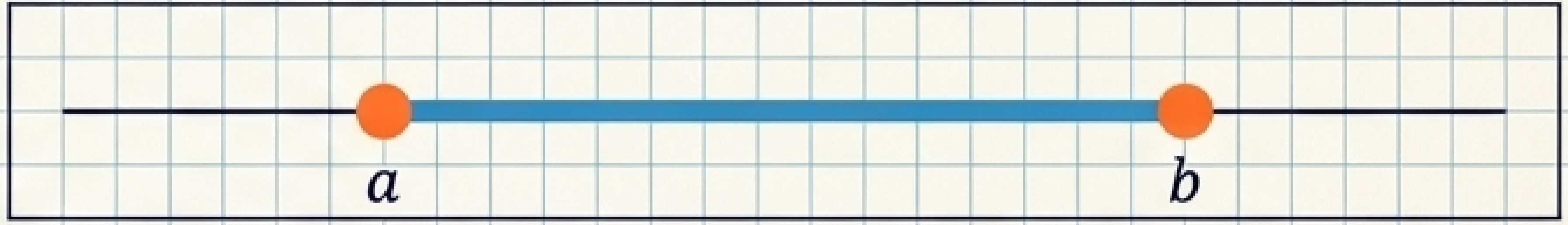
Intervals: Subsets on a Continuum

Core Insight: An interval is a bounded subset of real numbers containing infinitely many points. The length of any interval from a to b is $(b - a)$.

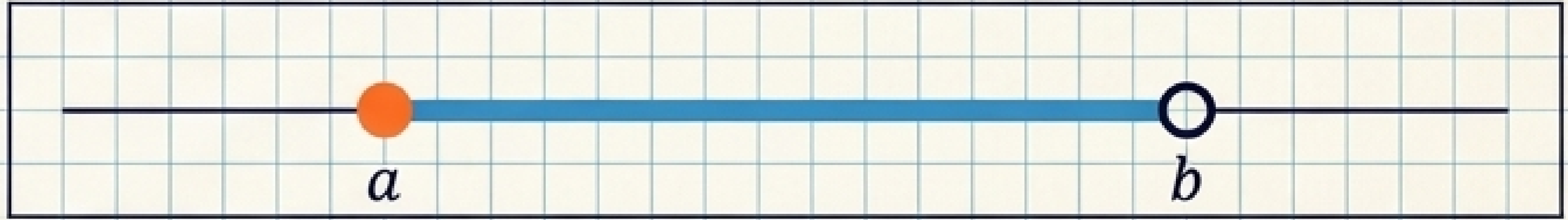
(a, b) - Open Interval.
$\{y : a < y < b\}$. Endpoints excluded.



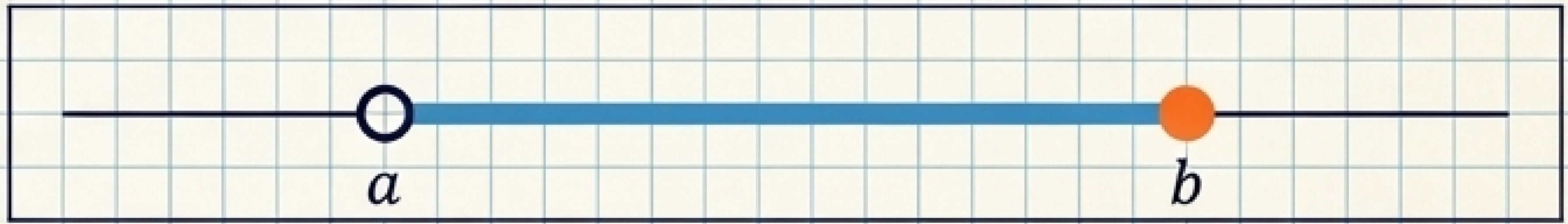
$[a, b]$ - Closed Interval.
$\{x : a \leq x \leq b\}$. Endpoints included.



$[a, b)$ - Half Open.
Included at a, excluded at b.



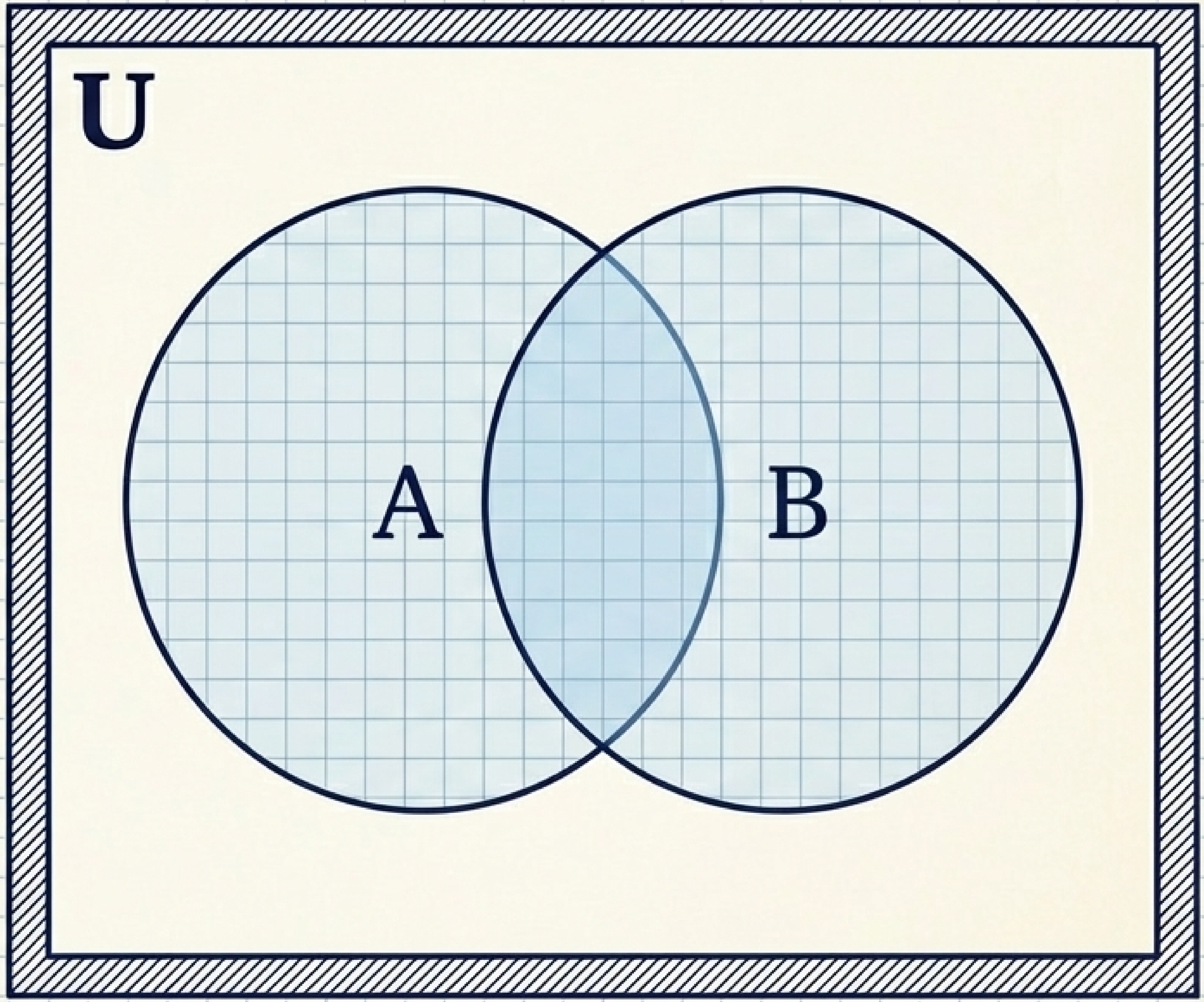
$(a, b]$ - Half Open.
Excluded at a, included at b.



Bounding the Context: The Universal Set & Venn Diagrams

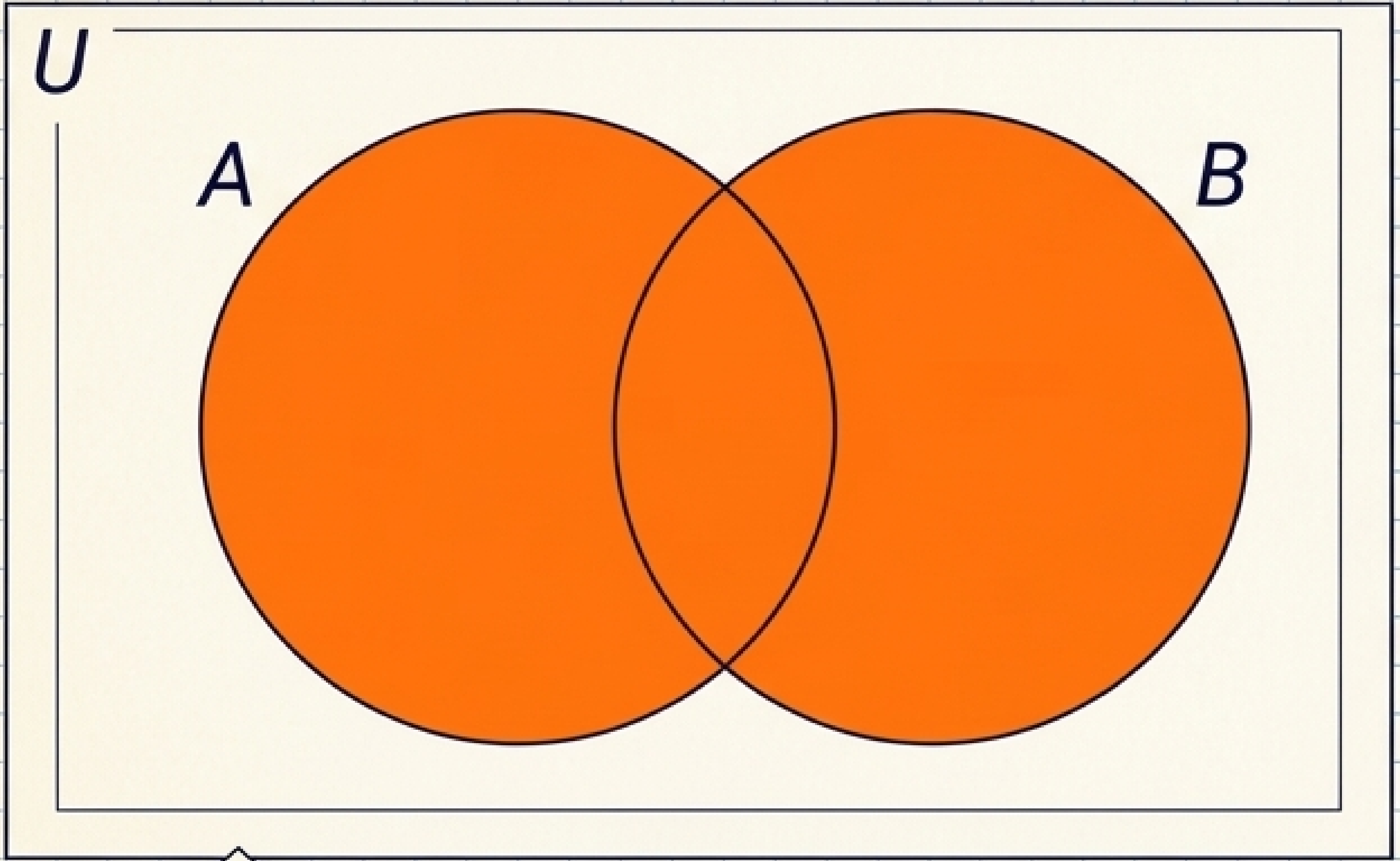
Core Insight: The Universal Set (U) is the absolute basic contextual boundary containing all objects under consideration (e.g., if analyzing a school, U = all students).

John Venn (1834-1883): Most relationships between sets can be mapped visually. Rectangles establish the universe (U); enclosed circles define the subsets. By layering these shapes, complex logic becomes instantly visible geometry.



Operation 1: Union ($A \cup B$)

The set of all elements which are in *A*, or in *B*, or in both. The logical OR.



Example: A = Hockey Team, B = Football Team.
$A \cup B$ = All student athletes playing either sport (or both).

Properties of Union:

Commutative:

$$A \cup B = B \cup A$$

Associative:

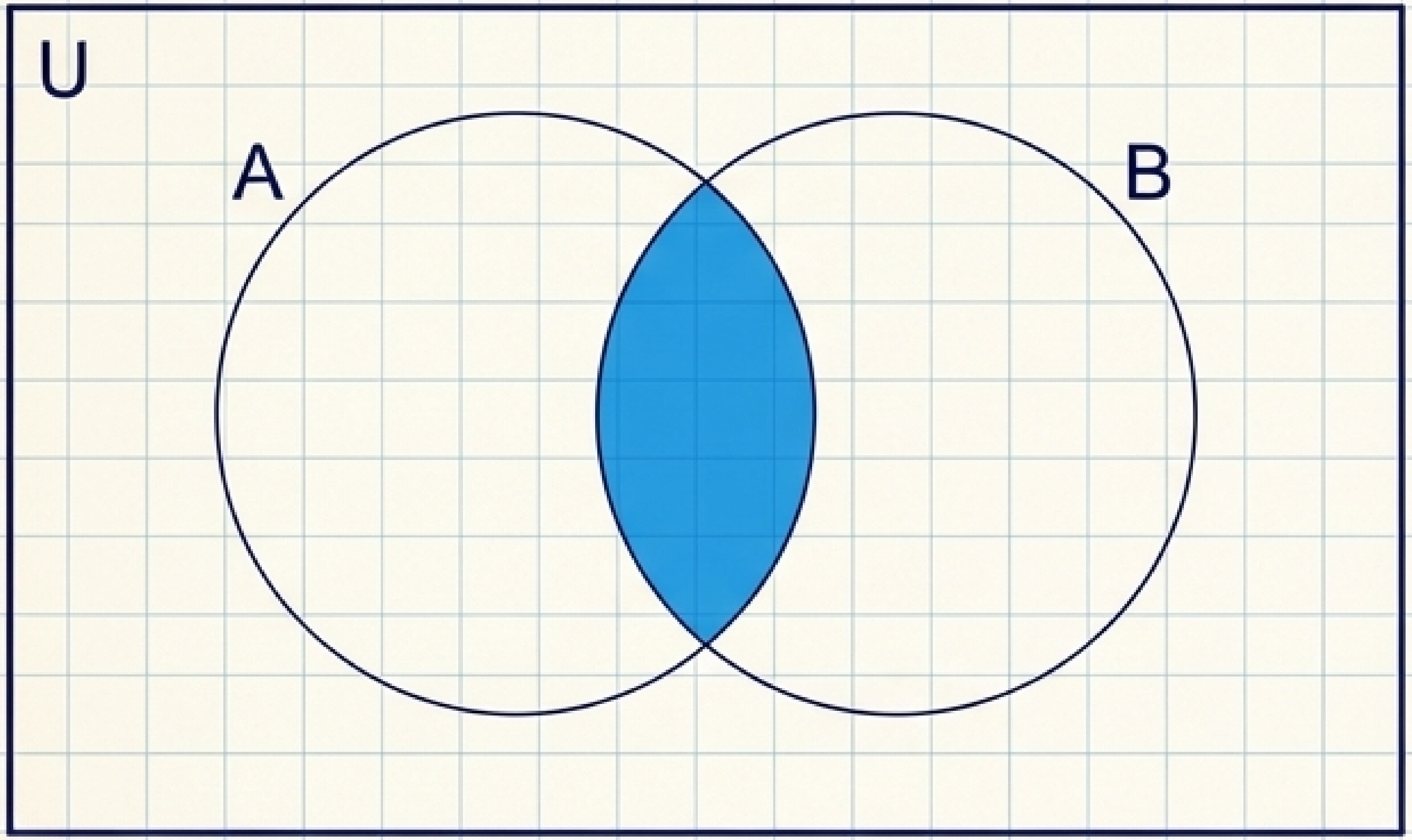
$$(A \cup B) \cup C = A \cup (B \cup C)$$

Law of Identity:

$$A \cup \emptyset = A$$

Operation 2: Intersection ($A \cap B$)

The set of all elements which are common to both A and B. The logical AND.

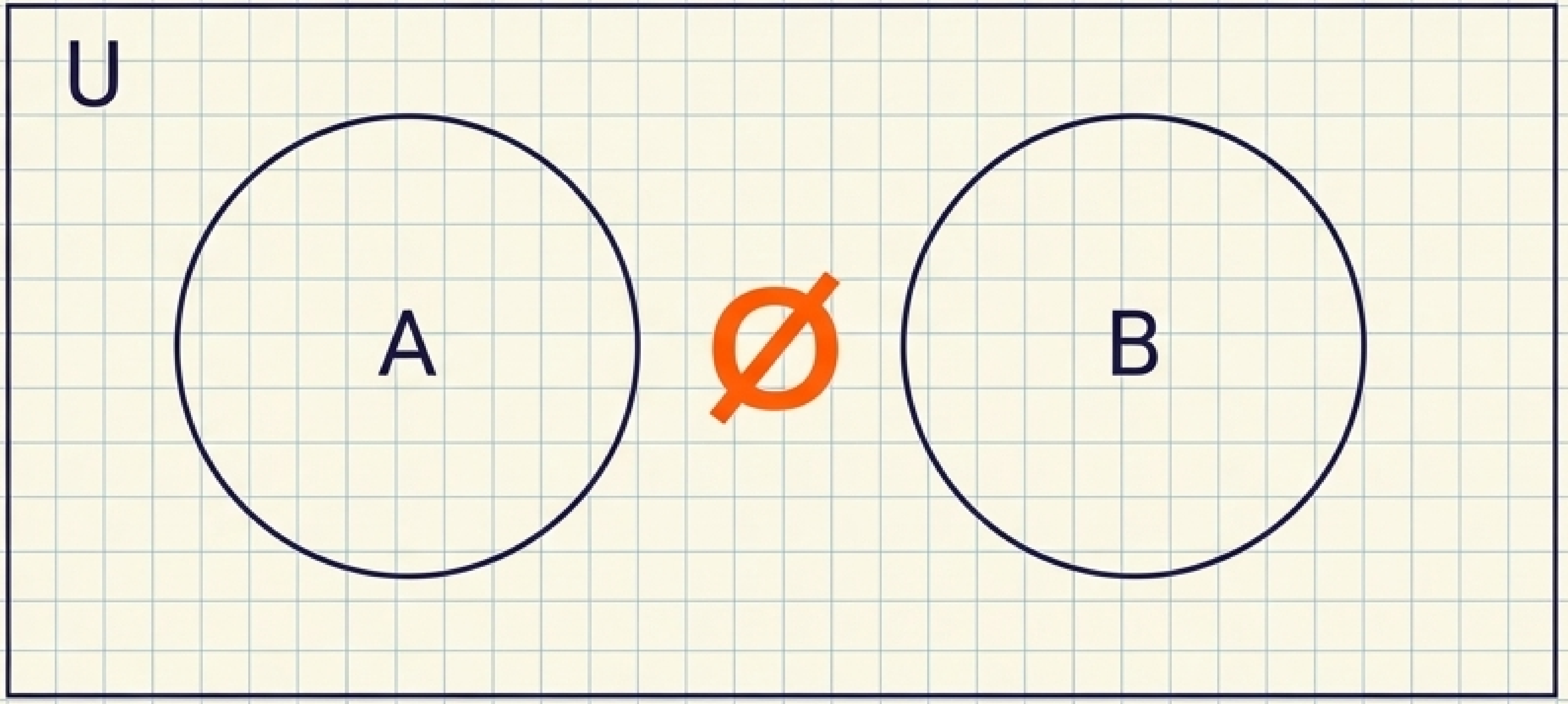


Example: A = Hockey Team, B = Football Team.
$A \cap B$ = The specific, dedicated students who play both sports simultaneously.

Properties of Intersection:	
Commutative:	$A \cap B = B \cap A$
Law of U:	$U \cap A = A$
Idempotent Law:	$A \cap A = A$

The Null Overlap: Disjoint Sets

If the intersection of two sets is the empty set $(A \cap B = \emptyset)$, the sets are entirely mutually exclusive. They share zero elements.

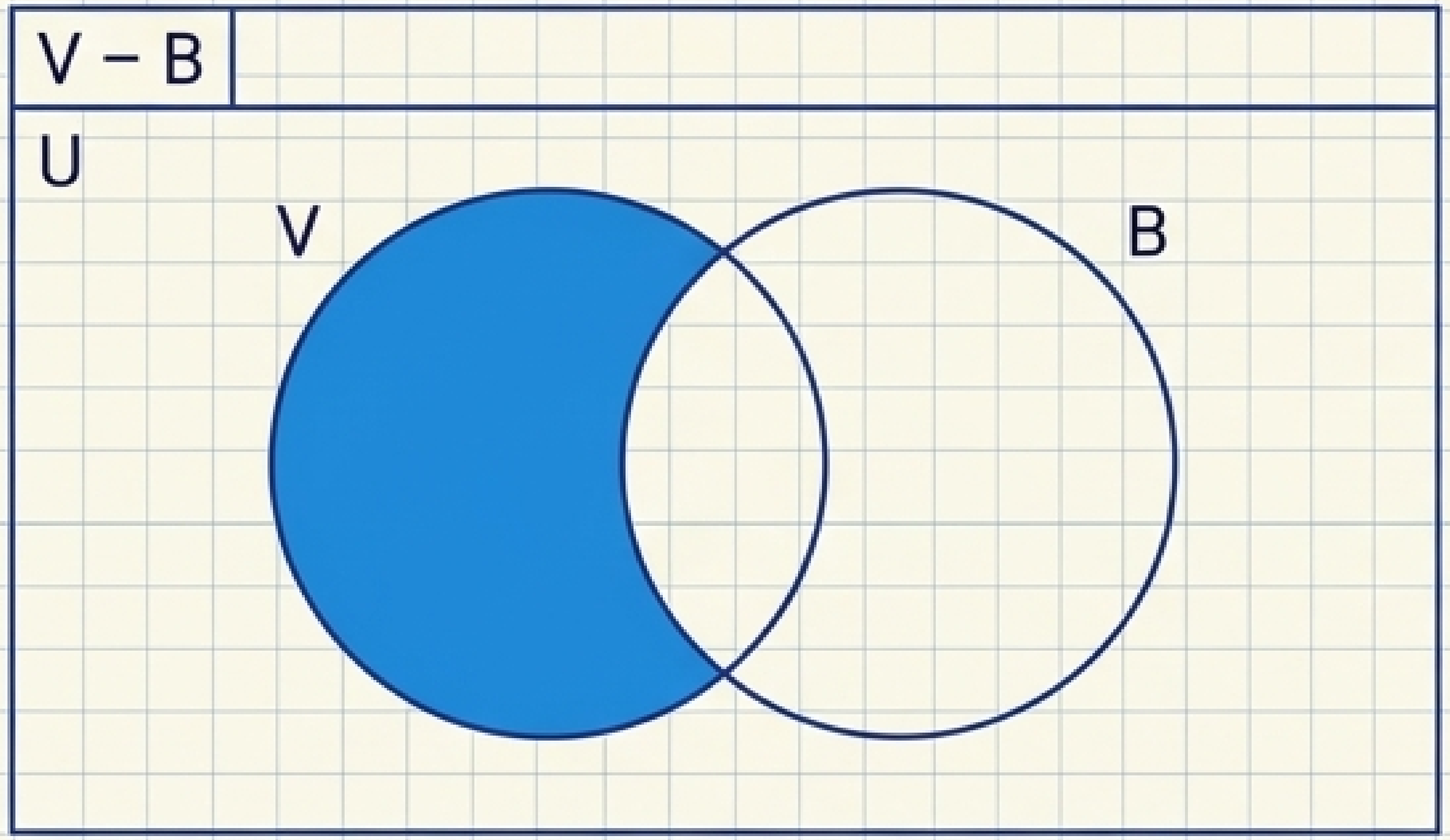


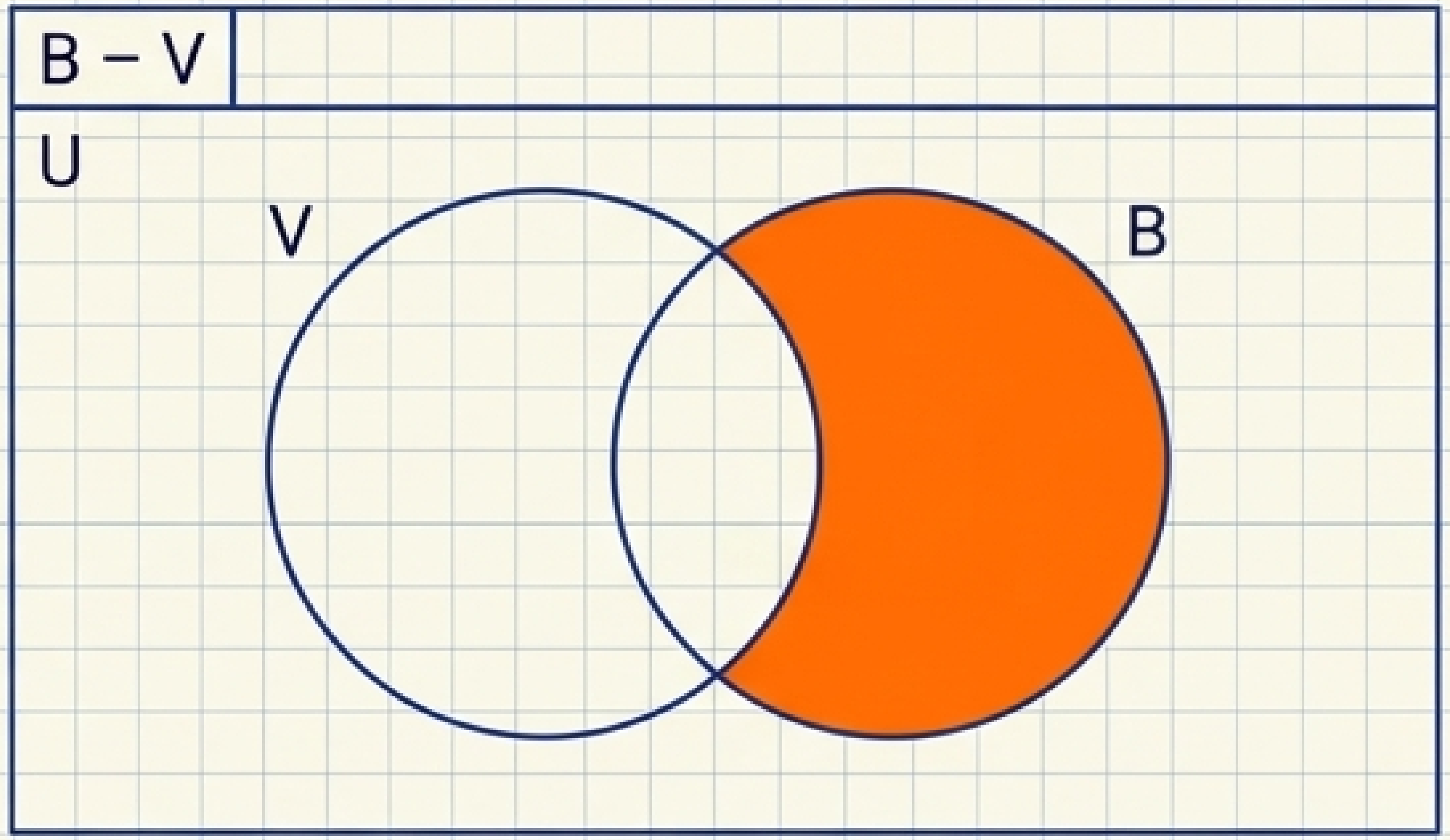
Set A = {2, 4, 6, 8} (Even Numbers)	Set B = {1, 3, 5, 7} (Odd Numbers)

Zero shared elements. They exist in total isolation.

Operation 3: Difference (−)

The set of elements that belong to A but not to B. The asymmetric logical NOT.





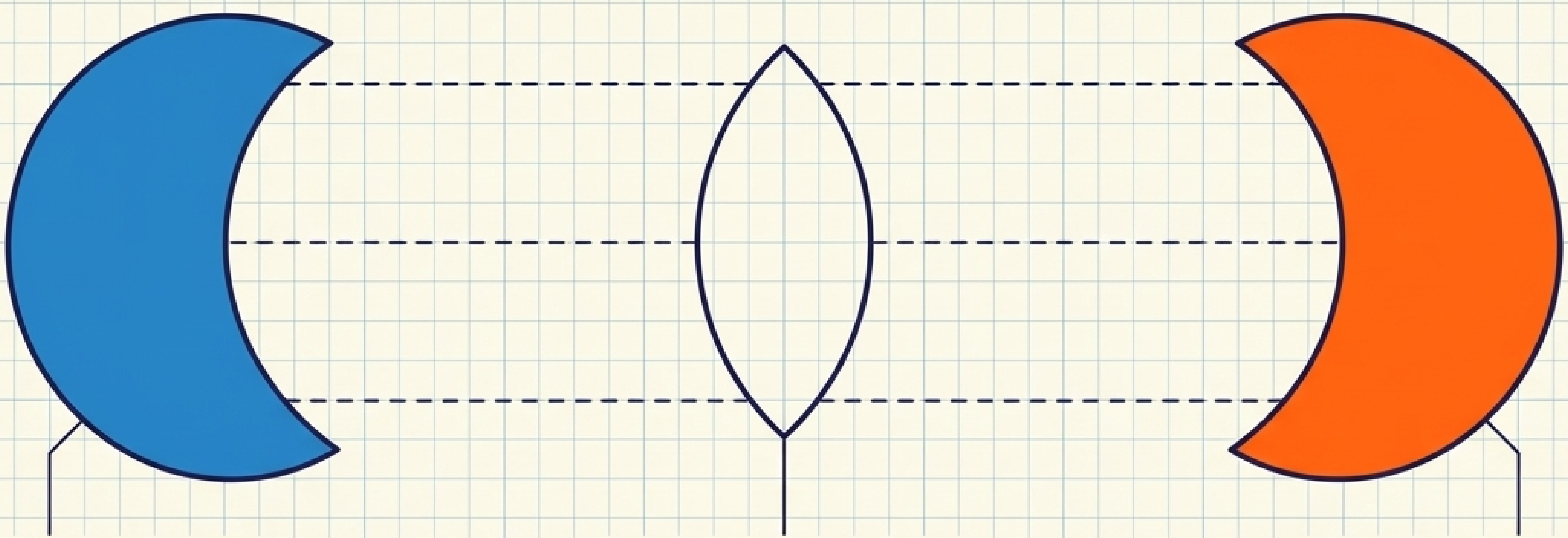
$V = \{a, e, i, o, u\}$
$B = \{a, i, k, u\}$
$V - B = \{e, o\}$

$B - V = \{k\}$

Crucial Insight:
Unlike Union and Intersection, Difference is NOT commutative.
$A - B \neq B - A$. It represents asymmetric surgical subtraction.

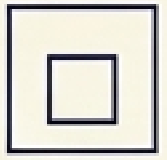
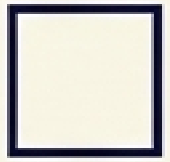
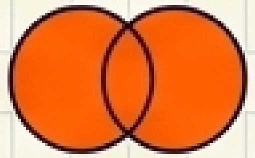
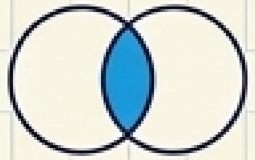
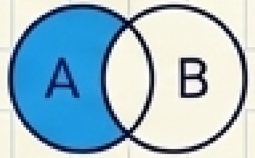
The Anatomy of Sets: Perfect Division

Any two intersecting sets can be perfectly exploded into three mutually disjoint sets. This reveals the true structural building blocks of Venn geometry.



None of these three pieces overlap. But when snapped together, they perfectly form the total Union (A ∪ ⊆ B). This is the ultimate elegance of set theory.

The Set Theory Toolkit

CONCEPT	SYMBOL	MEANING	VISUAL DIAGRAM
Element of	$\in$	Object belongs to a specific set.	
Subset	$\subset$	Every element of *A* is also in *B*.	
Empty Set	$\emptyset$	A set containing zero elements.	$\varnothing$
Universal Set	U	The total bounding context.	
Union	$\cup$	Elements in *A* **OR** *B*.	
Intersection	$\cap$	Elements in *A* **AND** *B*.	
Difference	$-$	Elements in *A* **NOT** in *B*.	

These foundational blueprints provide the exact language required to describe all modern mathematical reality.