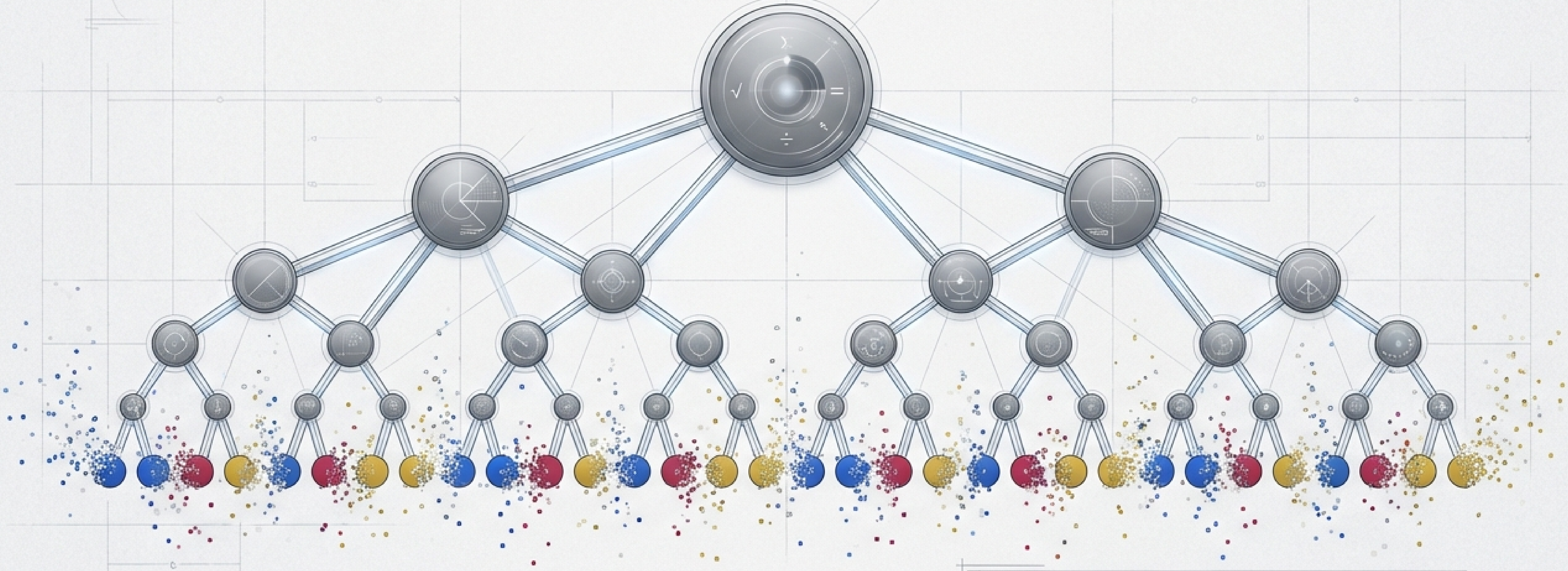
Real Numbers: The DNA of Mathematics

An exploration of the Fundamental Theorem of Arithmetic and the logic of Irrationality.



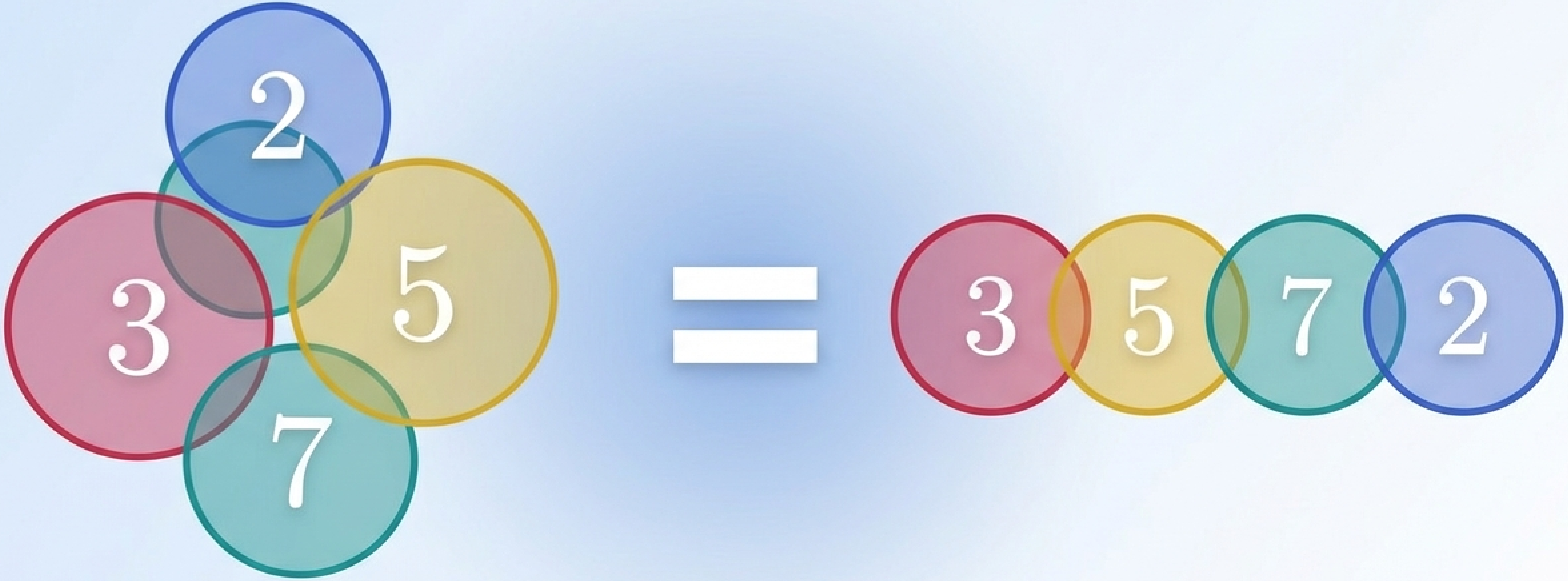
Just as matter is built from atoms, integers are built from primes.

Every Composite Number Has a Unique Signature

Theorem 1.1 (Fundamental Theorem of Arithmetic): Every composite number can be expressed as a product of primes, and this factorization is unique.

$$32760 = 2^3 \times 3^2 \times 5 \times 7 \times 13$$

Ingredients Define the Substance, Not the Mixing Order



The prime factorization of a natural number is unique, except for the order of its factors.

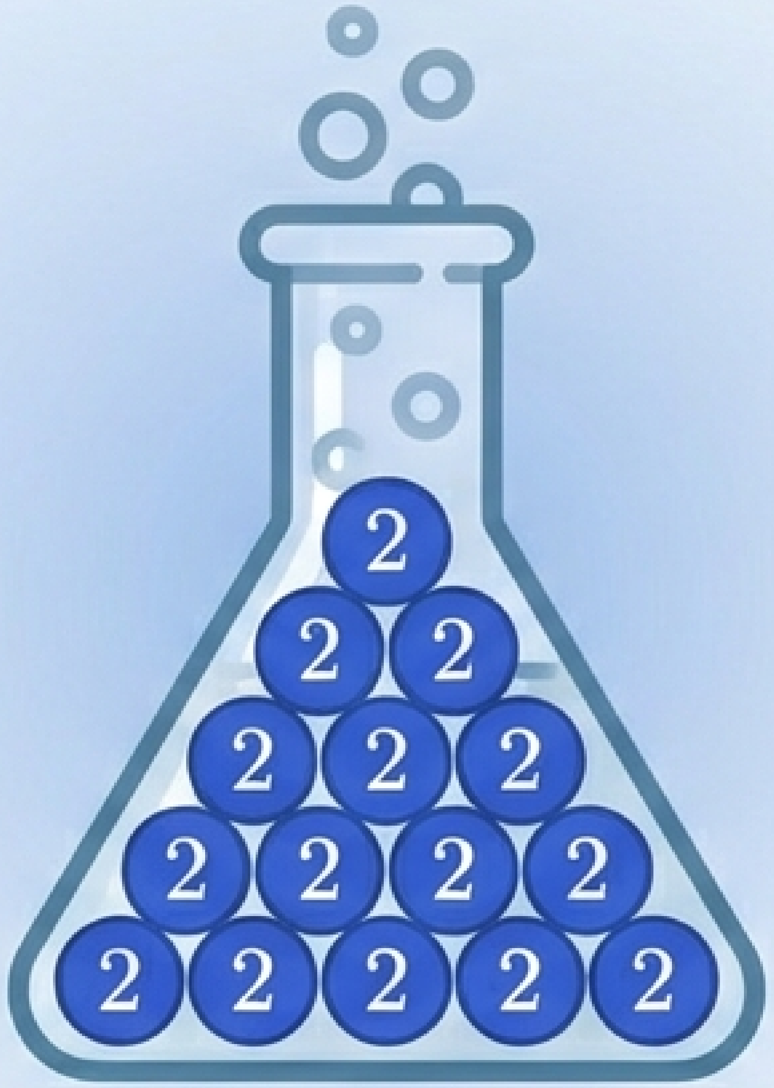
The “Digit Zero” Test

Can 4^n end in 0?

$$4^n$$

$$\downarrow$$

$$(2^2)^n = 2^{2n}$$



Prime Atoms of 4^n

Rule: To end in 0, a number must contain prime atoms of both 2 and 5. Since 5 is absent, 4^n never ends in zero.

Extracting the HCF and LCM

The Atoms

$$6 = 2^1 \times 3^1$$

$$20 = 2^2 \times 5^1$$

The Extraction

HCF (Highest Common Factor):
Product of smallest power of shared primes.

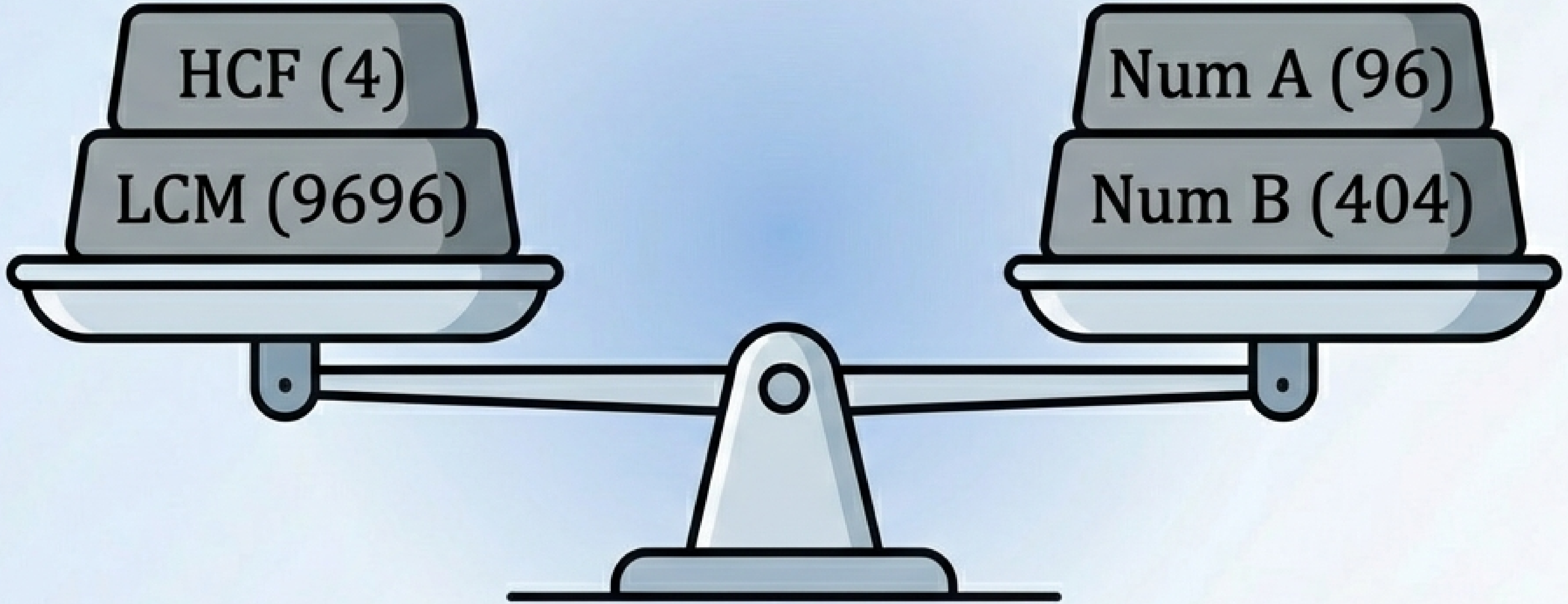
2^1 → Result: **2**

LCM (Lowest Common Multiple):
Product of greatest power of all primes.

$2^2, 3^1, 5^1$ → Result: $2^2 \times 3 \times 5 = \mathbf{60}$

The Golden Rule of Two Numbers

$$\mathrm{HCF}(a, b) \times \mathrm{LCM}(a, b) = a \times b$$



Verified: Both sides equal 38,784

Complexity Increases: The Exception for Three

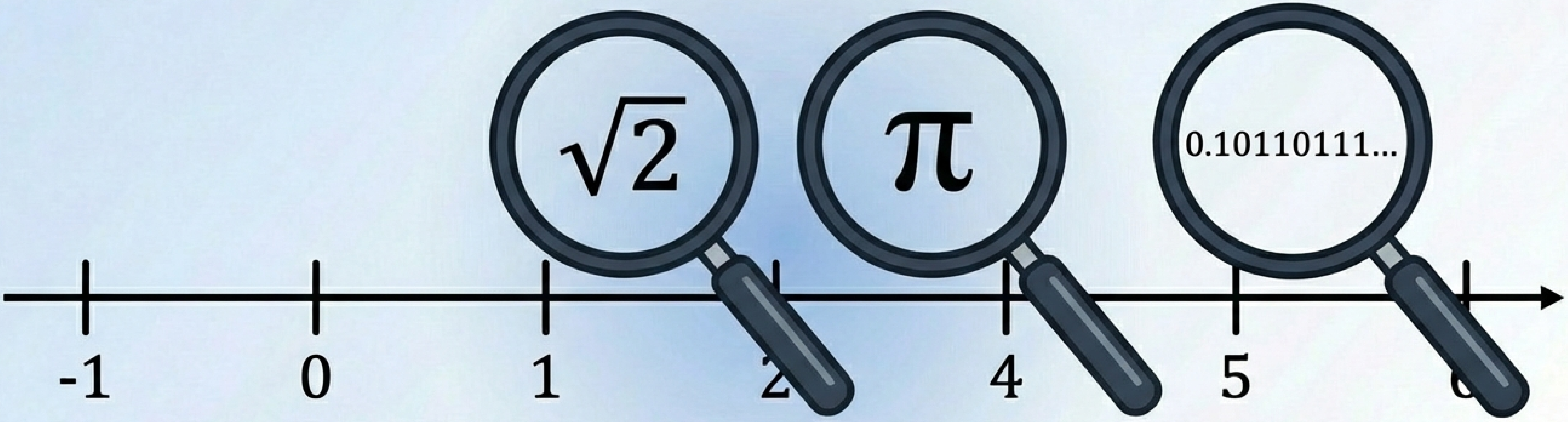
$$\text{Product of Numbers } (6 \times 72 \times 120) = 51{,}840$$

$$\neq$$

$$\text{HCF}(6) \times \text{LCM}(360) = 2{,}160$$

Remark: The product of three numbers is **NOT** equal to the product of their **HCF** and LCM.

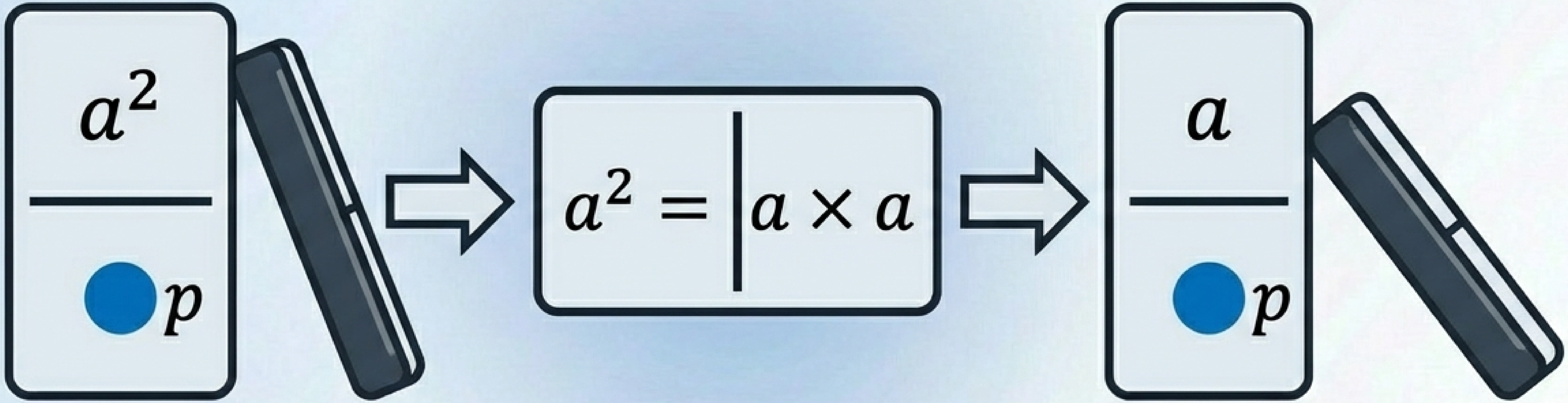
The Numbers That Don't Fit



Irrational Numbers: Cannot be written as p/q.
The gaps in the rational world.

The Tool of Contradiction: Theorem 1.2

If a prime divides the square, it divides the root.



Let p be a prime.
If p divides a^2, then p divides a.

Case Study: Proving $\sqrt{2}$ is Irrational (Part 1)

The Setup: Proof by Contradiction

Assumption: $\sqrt{2} = \frac{a}{b}$

We assume 'a' and 'b' share
NO common factors other than 1.

The Logic Loop (Part 2)

Square both sides: $2b^2 = a^2$

Therefore, 2 divides a

↓

Substitute $a = 2c$: $2b^2 = (2c)^2 \rightarrow 2b^2 = 4c^2$

↓

Simplify: $b^2 = 2c^2$

Therefore, 2 divides b

2 divides both $\boldsymbol{a}$ and $\boldsymbol{b}$. They are **not co-prime.**

Therefore, $\sqrt{2}$ cannot be rational.

A Universal Pattern for Primes

Case $\sqrt{2}$

$$2b^2 = a^2$$

Assumed co-prime, but 2 divides a

Case $\sqrt{3}$

$$3b^2 = a^2$$

Assumed co-prime, but 3 divides a

Case $\sqrt{5}$

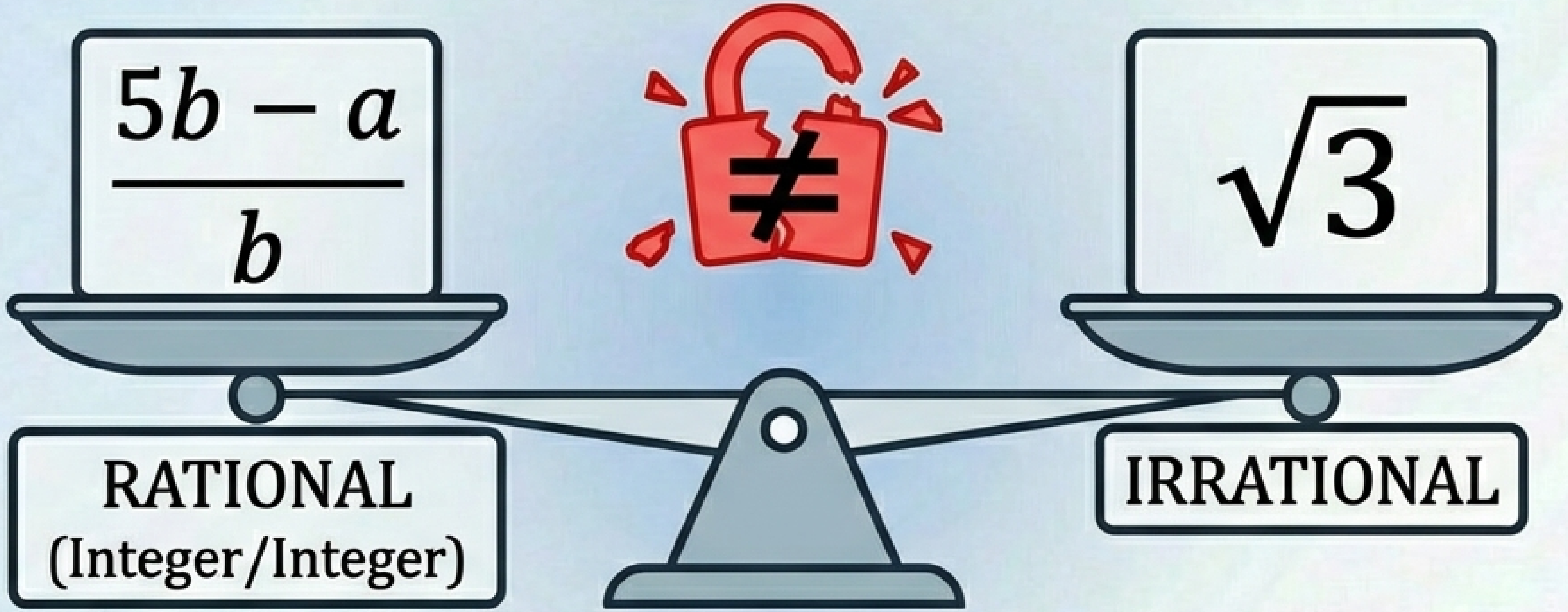
$$5b^2 = a^2$$

Assumed co-prime, but 5 divides a

The same contradiction arises for the square root of any prime number p.

Mixing Rationals and Irrationals

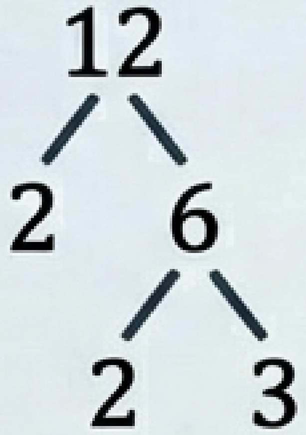
Is $5 - \sqrt{3}$ rational?



A Rational number cannot equal an Irrational number. The assumption fails.

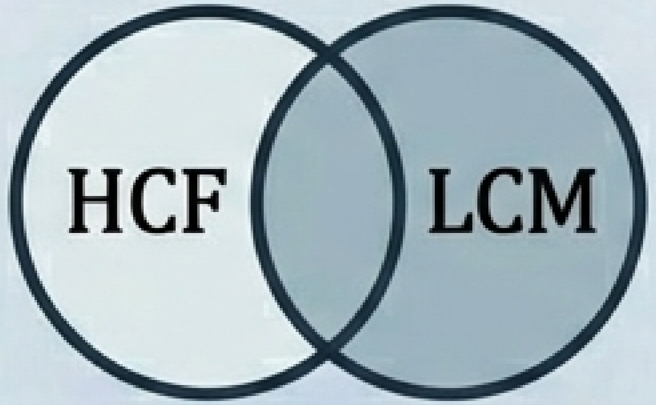
The Atomic Toolkit (Summary)

The Atoms



Every composite number is a unique product of primes.

The Mechanics



HCF = Smallest Powers.
LCM = Greatest Powers.

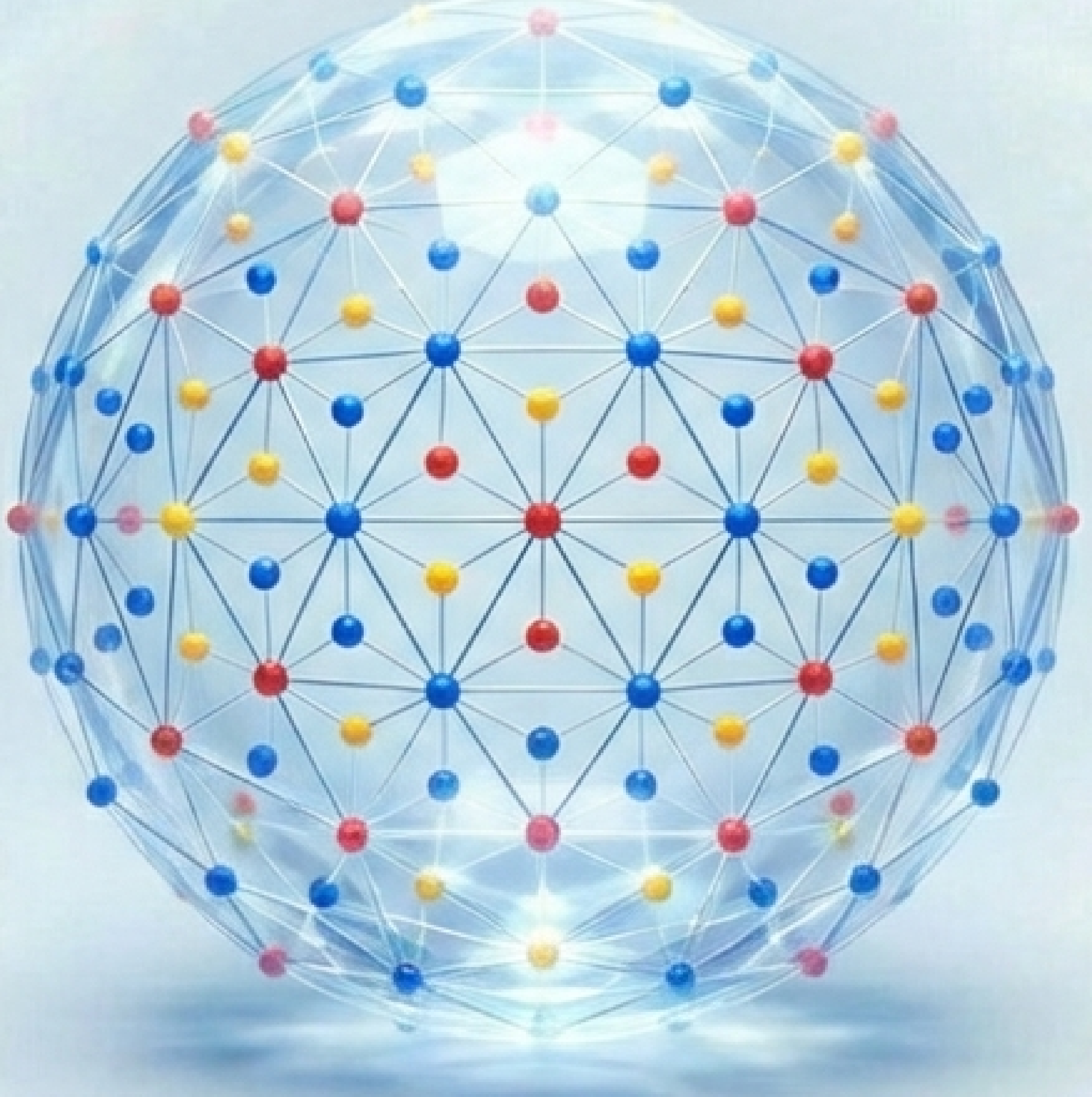
Product Rule applies to 2 numbers only.

The Proofs

Use contradiction.
If p divides a^2, p divides a.

Rationals mixed with Irrationals remain Irrational.

The Beauty of Unbreakable Rules



Mathematics relies on specific, unique building blocks. From the Fundamental Theorem to the proofs of irrationality, the logic holds because the ‘atoms’—the prime numbers—never change their nature.